Construction of Analytic Functions.

Milne-Thomson Method

Case (i):

If u is given, $f(z) = u+iv$

(i) Find u_x and u_y

ii) Find $u_x(z,0)$ and $u_y(z,0)$

iii) Find $f(z)$

$$= \int [u_x(z,0) - i\, u_y(z,0)]\, dz + c.$$

Case (ii):

If v is given,

$f(z) = u+iv$

i) Find v_x and v_y

ii) Find $v_x(z,0)$ and $v_y(z,0)$

(iii) Find $f(z)$

$$= \int [v_y(z,0) + i\, v_x(z,0)]\, dz + c$$

Problem:

1) ~~Prove that~~
$u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$
~~satisfies Laplace's eqn~~ and determine the corresponding analytic function
$f(z) = u + iv$

Soln:

Given; $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$

$f(z) = u + iv$.

Step-1: Find u_x & u_y

$u_x = 3x^2 - 3y^2 + 6x$

$u_y = -6xy - 6y$

Step-2: Find $u_x(z,0)$ and $u_y(z,0)$

$u_x(z,0) = 3z^2 - 3(0) + 6(z)$
$= 3z^2 + 6z$

$u_y(z,0) = -6(z)0 - 6(0)$
$= 0$.

Step-3:

$f(z) = \int [u_x(z,0) - i u_y(z,0)]\, dz + c$

$= \int [3z^2 + 6z - i(0)]\, dz + c$

$= \frac{3z^3}{3} + \frac{6z^2}{2} + c = z^3 + 3z^2 + c$

2) Find the analytic function whose Imaginary part is $e^{2x}[y\cos 2y + x\sin 2y]$

Soln:

Gn: $v = e^{2x}(y\cos 2y + x\sin 2y)$
$f(z) = u + iv$

Step-1

$v_x = e^{2x}[0 + \sin 2y] + [y\cos 2y + x\sin 2y]e^{2x}$

$v_x = e^{2x}\sin 2y + 2e^{2x}y\cos 2y + 2e^{2x}x\sin 2y$

$v_y = e^{2x}[y(-\sin 2y).2 + \cos 2y(1) + x\cos 2y.2]$

$= -e^{2x}2y\sin 2y + e^{2x}\cos 2y + 2x\cos 2y\, e^{2x}$

Step-2:

$v_x(z,0)$

$= 2e^{2z}(0) = 0$

$v_y(z,0)$

$= e^{2z}(1) + 2z(1)e^{2z}$

$= e^{2z} + 2ze^{2z}$

$= e^{2z}(1 + 2z)$

step-3:

$$f(z) = \int \left[V_y(z,0) + i V_x(z,0) \right] dz + c$$

$$= \int \left(e^{2z}(1+2z) + i(0) \right) dz + c$$

$$= \int e^{2z}(1+2z)\, dz + c$$

Bernoulli's Theorem

$$I = uv_1 - u'v_2 + u''v_3 - \cdots$$

$u = e^{2z}$ $dv = (1+2z)$

~~$f(z) = e^{2z}$~~

$u = e^{2z}$ $\qquad dv = (1+2z)$

$u' = e^{2z}.2$ $\qquad v_1 = z + 2\frac{z^2}{2}$

$u'' = 2.2.e^{2z}$ $\qquad v_2 = \frac{z^2}{2} + \frac{z^3}{3}$

$u''' = 4.2.e^{2z}$ $\qquad v_3 = \frac{z^3}{2.3} + \frac{z^4}{3.4}$

$\vdots$ $\qquad \vdots$

$$f(z) = e^{2x}(x+z^2)$$
$$- 2e^{2x}\left(\frac{z^2}{2}+\frac{z^3}{3}\right)$$
$$+ 4e^{2x}\left(\frac{z^3}{6}+\frac{z^4}{12}\right)$$
$$- \cdots$$

$$= ze^{2x} + \cancel{z^2 e^{2x}} - \cancel{z^2 e^{2x}}$$
$$- \frac{2}{3}\cancel{z^3 e^{2x}} + \frac{2}{3}\cancel{z^3 e^{2x}} + \frac{1}{3}z^4 e^{2z}$$
$$- \cdots$$

$$= ze^{2x} + \frac{1}{3}z^4 e^{2x} - \cdots + c.$$

$$\therefore f(z) = ze^{2x} + \frac{1}{3}z^4 e^{2x} - \cdots + c$$

---

3) Show that the function $u = x^3 + x^2 - 3xy^2 + 2xy - y^2$ is harmonic and hence find the analytic fun.

<u>Soln:</u>

Given:

$$u = x^3 + x^2 - 3xy^2 + 2xy - y^2$$

<u>Step-1</u>

$$u_x = 3x^2 + 2x - 3y^2 + 2y$$

$$u_{xx} = 6x + 2$$

$$u_y = -6xy + 2x - 2y$$

$$u_{yy} = -6x - 2$$

$$u_{xx} + u_{yy} = 0$$

$\therefore$ u is harmonic //

Step-2:

$u_x(z,0) = 3z^2 + 2z$

$u_y(z,0) = 2z$.

Step-3:

$$f(z) = \int \left[\frac{\partial u}{\partial x}(z,0) - i\frac{\partial u}{\partial y}(z,0)\right] dz$$

$$= \int 3z^2 + 2z - i(2z)\, dz$$

$$= \frac{3z^3}{3} + \frac{2z^2}{2} - i2\frac{z^2}{2} + c$$

$$f(z) = z^3 + z^2 - iz^2 + c.$$

is the required analytic function.

---

4) If $u = \dfrac{2\sin 2x}{e^{2y} + e^{-2y} - 2\cos 2x}$

find $w = f(z)$ such that $f(z)$ is analytic

Solution:

Gn : $u = \dfrac{2\sin 2x}{e^{2y} + e^{-2y} - 2\cos 2x}$ —(1)

Step-1

formula : $\cosh ay = \dfrac{e^{ay} + e^{-ay}}{2}$

$2\cosh ay = e^{ay} + e^{-ay}$

$$u = \frac{2\sin 2x}{2\cosh 2y - 2\cos 2x}$$

$$= \frac{\sin 2x}{\cosh 2y - \cos 2x}$$

Step 1:

$$u_x = \frac{(\cosh 2y - \cos 2x)(\cos 2x) - \sin 2x\,(+\sin 2x)}{(\cosh 2y - \cos 2x)^2}$$

$$= \frac{2\cos 2x \cosh 2y - 2\cos^2 x - 2\sin^2 x}{(\cosh 2y - \cos 2x)^2}$$

$$= \frac{2\cos 2x \cosh 2y - 2(1)}{(\cosh 2y - \cos 2x)^2}$$

$$= \frac{2\cos 2x \cosh 2y - 2}{(\cosh 2y - \cos 2x)^2}$$

Step-2

$u_x(z,0)$

$$= \frac{2\cos 2z \cosh(0) - 2}{(1 - \cos 2z)^2}$$

$$= \frac{2\cos 2z(1) - 2}{(1 - \cos 2z)^2}$$

$$= \frac{-2(1 - \cos 2z)}{(1 - \cos 2z)^2}$$

$$= \frac{-2}{(1 - \cos 2z)}$$

$(\cos 2\theta = 1 - 2\sin^2\theta)$

Step-2

$$u_x(x,0) = \frac{-2}{1-(1-2\sin^2 x)}$$

$$= \frac{-2}{+2\sin^2 x}$$

$$= \frac{-1}{\sin^2 x}$$

$$u_x(x,0) = -\operatorname{cosec}^2 x$$

Step-3

$$f(x) = \int \left[\frac{\partial u}{\partial x}(x,0) - i\frac{\partial u}{\partial y}(x,0)\right] dx$$

$$= \int -\operatorname{cosec}^2 x \, dx$$

$$f(x) = \cot x + c.$$

To find u_y

$u_y(x,0)$

Step-1 u_y

$$u = \sin 2x \, [\cosh 2y - \cos 2x]^{-1}$$

$$u_y = -\sin 2x \, [\cosh 2y - \cos 2x]^{-2} \, [-\sinh 2y \cdot 2]$$

$$= \frac{2\sin 2x \sinh 2y}{[\cosh 2y - \cos 2x]^2}$$

Step-2

$$u_y(x,0) = \frac{2\sin 2x \sinh 2(0)}{[\cosh 2(0) - \cos 2x]^2}$$

$$= \frac{2\sin 2x\,(0)}{[1-\cos 2x]^2}$$

$$u_y(x,0) = 0$$

Scanned by CamScanner