Cauchy's Riemann equation (Polar coordinates)

If the function $w = f(z) = u(r,\theta) + iv(r,\theta)$ is analytic in region R of the z-plane, then

i) $\frac{\partial u}{\partial r}, \frac{\partial u}{\partial \theta}, \frac{\partial v}{\partial r}, \frac{\partial v}{\partial \theta}$ exists.

ii) $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$ and $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$

Problems:

Prove that $w = z^2$ is analytic.

Sol

Given $w = z^2$

WKT $z = x+iy \Rightarrow z^2 = (x+iy)^2 = x^2 - y^2 + 2ixy$

$f(z) = u(x,y) + iv(x,y)$

$w = f(z) = u(x,y) + iv(x,y) = x^2 - y^2 + 2ixy$

$u = x^2 - y^2$, $v = 2xy$

$\frac{\partial u}{\partial x} = 2x$ $\qquad$ $\frac{\partial v}{\partial x} = 2y$

$\frac{\partial u}{\partial y} = -2y$ $\qquad$ $\frac{\partial v}{\partial y} = 2x$

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$\therefore$ $w = z^2$ is analytic.

2. Determine whether the function $w = 2xy + i(x^2 - y^2)$ is analytic

Sol

$w = 2xy + i(x^2 - y^2)$

$f(z) = u(x,y) + iv(x,y)$.

$u = 2xy$, $\frac{\partial u}{\partial x} = 2y$, $\frac{\partial u}{\partial y} = 2x$

$v = x^2 - y^2$ $\quad$ $\frac{\partial v}{\partial x} = 2x$ $\quad$ $\frac{\partial v}{\partial y} = -2y$

$$\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial u}{\partial y} \neq -\frac{\partial v}{\partial x}$$

$\therefore$ The given function is not analytic

3. Verify whether the function $f(z) = e^{-x}(\cos y - i\sin y)$ is analytic justify

$$f(z) = e^{-x}\cos y - i e^{-x}\sin y$$

$$f(z) = u(x, y) + iv(x, y)$$

$$u = e^{-x}\cos y \qquad v = -e^{-x}\sin y$$

$$\frac{\partial u}{\partial x} = -e^{-x}\cos y \qquad \frac{\partial v}{\partial x} = e^{-x}\sin y$$

$$\frac{\partial u}{\partial y} = e^{-x}\sin y \qquad \frac{\partial v}{\partial y} = -e^{-x}\cos y$$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$\therefore$ It is analytic