STOKE'S THEOREM

Statement

If $\vec{F}$ is any continuous differentiable vector function and S is a surface enclosed by a curve C, then $\int_C \vec{F}\cdot d\vec{r} = \iint_S \nabla\times\vec{F}\cdot\hat{n}\, ds$

where $\hat{n}$ is the unit normal vector at any point of S.

Note:

1) If $\vec{F}$ is irrotational $\nabla\times\vec{F}=0$

$\int_C \vec{F}\, dr = \iint_S \nabla\times\vec{F}\,\hat{n}\,ds = 0$ and hence $\vec{F}$ is conservative

2) Let $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$

$$\int_C P\,dx + Q\,dy + R\,dz = \iint_S \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) dy\,dz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) dz\,dx + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

Problems

① Evaluate $\int_C \vec{F}\cdot d\vec{r}$ by stoke's theorem where $\vec{F} = y^2\vec{i} + x^2\vec{j} - (x+z)\vec{k}$ and C is the boundary of the triangle with vertices at $(0,0,0)$, $(1,0,0)$, $(1,1,0)$

Sol

By stoke's theorem

$$\int_C \vec{F}\,d\vec{r} = \iint_S \nabla\times\vec{F}\cdot\hat{n}\,ds$$

Since z coordinate is zero in all three vertices of the given triangle, the triangle lies on the xy plane

$$\vec{F} = y^2\vec{i} + x^2\vec{j} - (x+z)\vec{k}$$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^2 & x^2 & -(x+z) \end{vmatrix}$$

$$= \vec{i}(0) - \vec{j}(-1+0) + \vec{k}(2x-2y)$$

$$= \vec{j} + 2(x-y)\vec{k}$$

Since the triangle lies on xy plane and hence the unit ~~normal~~ vector normal to the surface OAB is $\vec{k}$

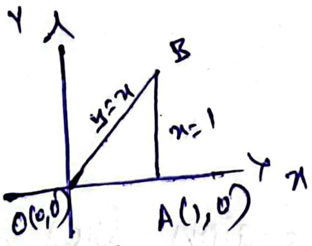
ie) $\hat{n} = \vec{k}$

$$\nabla \times \vec{F} \cdot \hat{n} = [\vec{j} + 2(x-y)\vec{k}] \cdot \vec{k}$$

$$= 2(x-y)$$

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S \nabla \times \vec{F} \cdot \hat{n}\, ds$$

$$= 2\int_0^1\int_y^1 (x-y)\, dx\, dy$$



$$= 2\int_0^1 \left[\frac{x^2}{2} - yx\right]_y^1 dy$$

$$= 2\int_0^1 \left(\frac{1}{2} - y - \frac{y^2}{2} + y^2\right) dy$$

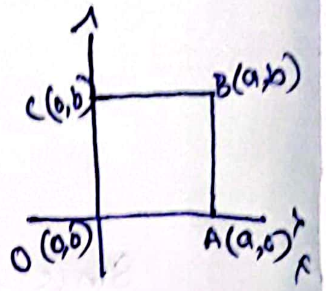
$$= 2\left[\frac{y}{2} - \frac{y^2}{2} - \frac{y^3}{6} + \frac{y^3}{3}\right]_0^1$$

$$= 2\left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{2}} - \frac{1}{6} + \frac{1}{3}\right)$$

$$= -\frac{1}{3} + \frac{2}{3} = \frac{1}{3}$$

② Verify Stoke's theorem for the vector field by $\vec{F} = (x^2-y^2)\vec{i} + 2xy\vec{j}$ in the rectangular region xy plane bounded by the lines

$x=0, x=a, y=0, y=b$



Sol

Given $\vec{F} = (x^2-y^2)\vec{i} + 2xy\vec{j}$

By Stoke's theorem

$$\iint_S \vec{F}\cdot dr = \iint_S \nabla\times\vec{F}\cdot\hat{n}\,ds$$

R.H.S ⇒

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2-y^2 & 2xy & 0 \end{vmatrix}$$

$$= \vec{i}(0) - \vec{j}(0) + \vec{k}\left(\frac{\partial}{\partial x}(2xy) - \frac{\partial}{\partial y}(x^2-y^2)\right)$$

$$= \vec{k}\,(2y+2y)$$

$$= 4y\,\vec{k}$$

Since the rectangular region lies on xy plane & hence unit vector normal to the surface OABC is $\vec{k}$ $\therefore \nabla\times\vec{F}\cdot\hat{n} = 4y$

$$\iint_S \nabla\times\vec{F}\cdot\hat{n}\,ds = \int_0^b\int_0^a 4y\,dx\,dy$$

$$= \int_0^b (4xy)_0^a\,dy$$

$$= \int_0^b (4ay)\,dy$$

$$= \left[\frac{4ay^2}{2}\right]_0^b = 2ab^2$$

R.H.S

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$d\vec{r} = dx\,\vec{i} + dy\,\vec{j} + dz\,\vec{k}$$

$$\vec{F}\cdot d\vec{r} = [(x^2-y^2)\vec{i} + 2xy\,\vec{j} + 0\vec{k}]$$
$$[dx\,\vec{i} + dy\,\vec{j} + dz\,\vec{k}]$$

$$= (x^2-y^2)\,dx + 2xy\,dy$$

$$\int_C \vec{F}\,d\vec{r} = \int_{OA} + \int_{AB} + \int_{BC} + \int_{CD}$$

$$\int_{OA} \vec{F}\cdot d\vec{r} = \int_0^a (x^2-y^2)\,dx + 2xy\,dy$$

$$= \int_0^a x^2\,dx \qquad y=0,\ dy=0$$

$$= x^3/3 = a^3/3$$

$$\int_{AB} \vec{F}\cdot dr = \int_{AB} (x^2-y^2)\,dx + 2xy\,dy$$

$$= \int_0^b 0 + 2ay\,dy \qquad x=a,\ dx=0$$

$$= \left[\frac{\cancel{2}ay^2}{\cancel{2}}\right]_0^b = ab^2$$

$$\int_{BC} \vec{F}\cdot d\vec{r} = \int_a^0 (x^2-y^2)\,dx + 2xy\,dy \qquad y=b,\ dy=0$$

$$= \int_a^0 (x^2-b^2)\,dx$$

$$= \left[x^3/3 - xb^2\right]_a^0$$

$$= -a^3/3 + ab^2$$

$$\int_{CO} \vec{F}\, d\vec{r} = \int_{CO} (x^2 - y^2)\, dx + 2xy\, dy$$

$x = 0$

$dx = 0$

$$= 0 + 0 = 0$$

$$\int_C \vec{F} \cdot d\vec{r} = a^3/3 + ab^2 - a^3/3 + ab^2$$

$$= 2ab^2$$

L.H.S = R.H.S

Hence Proved.