Applications of Double Integral (Area)

$$\text{area} = \iint dx\,dy \quad (\text{or}) \quad \iint dy\,dx$$

1. Evaluate $\iint dx\,dy$ over the region bounded by $x=0$, $x=2$, $y=0$, $y=2$

Sol

$$\text{Area} = \iint dx\,dy$$
$$= \int_0^2\int_0^2 dx\,dy$$
$$= \int_0^2 [x]_0^2\, dy$$
$$= 2\int_0^2 dy = 2(2)$$
$$= 4$$

2. Evaluate $\iint dx\,dy$, where R is the shaded region in the figure

Sol

Area of the shaded region

$= \text{Area of semicircle}$

$= \frac{1}{2}\pi r^2$

$= \frac{1}{2}\pi(2)^2 = \frac{4\pi}{2}$

$= 2\pi$

3. Find the area between the curves $y^2 = 4x$ and $x^2 = 4y$

Sol

$$\text{Area} = \iint dy\,dx$$

$y^2 = 4x \rightarrow (1)$

$x^2 = 4y$

$\frac{x^2}{4} = y \rightarrow (2)$

Sub eqn (2) in (1)

$(1) \Rightarrow y^2 = 4x$

$\left(\frac{x^2}{4}\right)^2 = 4x$

$x^3 = 64$

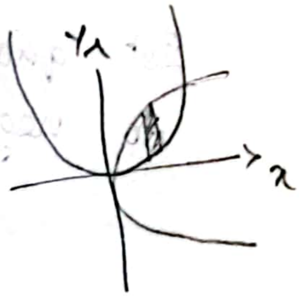
$\boxed{x = 4}$

Sub $x=4$ in eqn (1)

$$y^2 = 4(4)$$
$$= 16$$
$$\boxed{y = 4}$$

x limit $x=0$ to $x=4$

y limit $y = \frac{x^2}{4}$ to $y = 2\sqrt{x}$



$$A = \int_0^4 \int_{x^2/4}^{2\sqrt{x}} dy\, dx$$

$$= \int_0^4 [y]_{x^2/4}^{2\sqrt{x}} dx$$

$$= \int_0^4 \left[2\sqrt{x} - \frac{x^2}{4}\right] dx$$

$$= \frac{1}{4} \int_0^4 (8\sqrt{x} - x^2)\, dx$$

$$= \frac{1}{4} \left[\frac{8x^{3/}}{3/} - \frac{x^3}{3}\right]_0^4$$

$$= \frac{1}{4} \left[\frac{x^{3/}}{3} - \frac{x^3}{3}\right]_0^4$$

$$= \frac{1}{4} \left(\frac{16 \times 4\sqrt{4}}{3} - \frac{64}{3}\right)$$

$$= \left(\frac{16 \times 2}{3} - \frac{16}{3}\right)$$

$$= 16/3$$

4. Find the area bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Sol

Given $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$\text{Area} = \iint_R dy\, dx$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$$

$$\frac{y^2}{b^2} = \frac{a^2-x^2}{a^2}$$

$$y^2 = \frac{b^2}{a^2}(a^2-x^2)$$

$$y = \pm \frac{b}{a}\sqrt{a^2-x^2}$$

Ist quadrant will always comes with possitive so negative is not considered.

$$y = \frac{b}{a}\sqrt{a^2-x^2}$$

limits x limit : $x=0$ to $x=a$

y limit : $y=0$ to $y=\frac{b}{a}\sqrt{a^2-x^2}$

Area of the ellipse = 4 × Area of Ist quadrant.

$$= 4\int_0^a \int_0^{b/a\sqrt{a^2-x^2}} dy\, dx$$

$$= 4\int_0^a [y]_0^{b/a\sqrt{a^2-x^2}} dx$$

$$= 4\int_0^a \frac{b}{a}\sqrt{a^2-x^2}\, dx = \frac{4b}{a}\int_0^a \sqrt{a^2-x^2}\, dx$$

$$= \frac{4b}{a}\left[\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]_0^a$$

$$= \frac{4b}{a}\left[\frac{a}{2}\sqrt{a^2-a^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{a}{a}\right)\right]$$

$$= \frac{4b}{a}\left[\frac{a^2}{2} \times \frac{\pi}{2}\right]$$

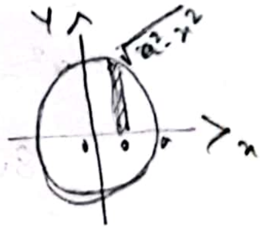
$$= \pi ab$$

$\therefore$ Area of ellipse $= \pi ab$

5. Find the area of the circle of radius a by double integration

Sol

Equation of circle $x^2+y^2=a^2$



Area $= \iint dy\, dx$

$x^2+y^2=a^2$

$y^2=a^2-x^2$

$y=\pm\sqrt{a^2-x^2}$

1st Quadrant always will be positive.

$y=\sqrt{a^2-x^2}$

limits x limit : $x=0$ to $x=a$

y limit : $y=0$ to $y=\sqrt{a^2-x^2}$

Area of circle $= 4 \times$ Area of I quadrant

$= 4\int_0^a\int_0^{\sqrt{a^2-x^2}} dy\, dx$

$= 4\int_0^a [y]_0^{\sqrt{a^2-x^2}}\, dx$

$= 4\int_0^a \sqrt{a^2-x^2}\, dx$

$= 4\left[\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]_0^a$

$= 4\left[\frac{a}{2}\sqrt{a^2-a^2}+\frac{a^2}{2}\sin^{-1}(1)\right]$

$= 2a^2\frac{\pi}{2} = \pi a^2$

Area of circle $= \pi a^2$

6. Using double integration find the area of enclosed by the curves $y=2x^2$, $y^2=4x$

Sol

$y=2x^2$, $y^2=4x$

$x^2=\frac{y}{2} \rightarrow (1)$ $x=\frac{y^2}{4} \rightarrow (2)$

Sub (2) in (1)

$\left(\frac{y^2}{4}\right)^2=\frac{y}{2}$

$\frac{y^4}{16}=\frac{y}{2}$

$y^3=8$

$\boxed{y=2}$

Sub $y=2$ in (1)

(1) $\Rightarrow$ $x^2=\frac{y}{2}$

$x^2=\frac{2}{2}$

$\boxed{x=1}$

$\therefore$ x limit $x=0$ to $x=1$

y limit $y=2x^2$ to $y=2\sqrt{x}$

$$I=\int_0^1\int_{2x^2}^{2\sqrt{x}} dy\,dx$$

$$=\int_0^1 [y]_{2x^2}^{2\sqrt{x}}\,dx$$

$$=\int_0^1 (2\sqrt{x}-2x^2)\,dx$$

$$=2\left[\frac{x^{3/2}}{3/2}-\frac{x^3}{3}\right]_0^1$$

$$=\frac{4}{3}-\frac{2}{3}=\frac{2}{3} //.$$

TRIPLE INTEGRATION IN CARTESIAN COORDINATES

Triple integration of a function defined over a region $\iiint f(x,y,z)\,dx\,dy\,dz$

Note:

$\iiint_R dx\,dy\,dz \rightarrow$ volume of the region.