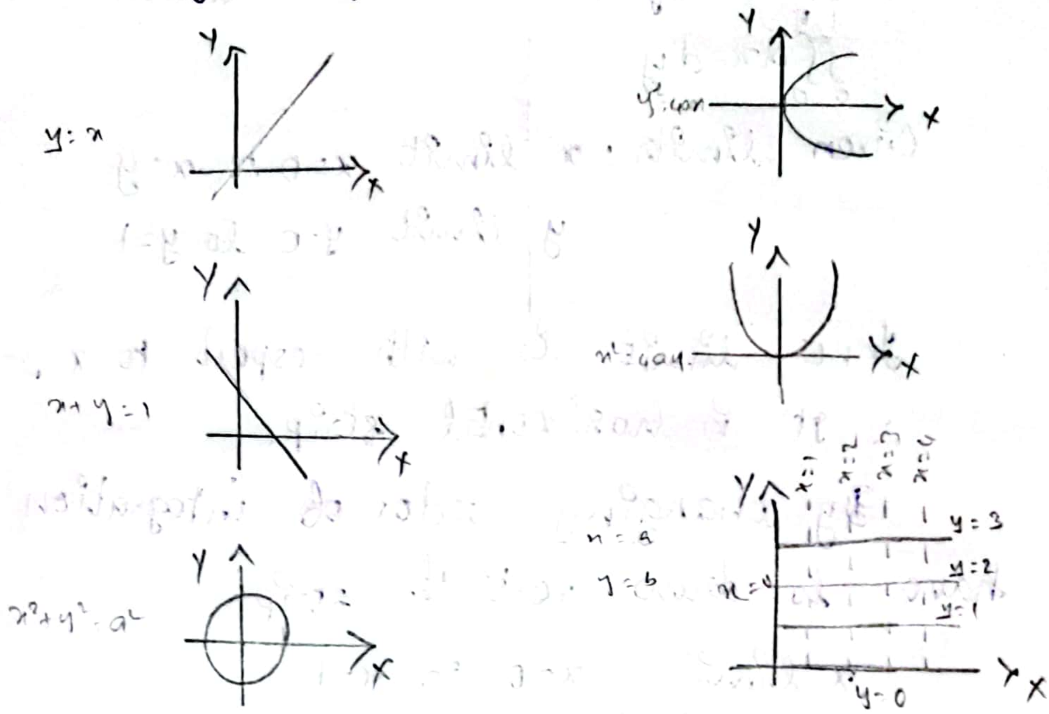
Double Integration : Under given region

Standard diagrams :



Change of order of integration

1. Change of order of integration for $\int_0^1\int_0^x dx\,dy$

Sol

Given integral is not in the correct form

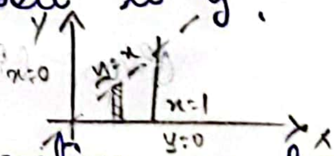
$\int_0^1\int_0^x dy\,dx$

Given limits x limit $x=0$ to $x=1$

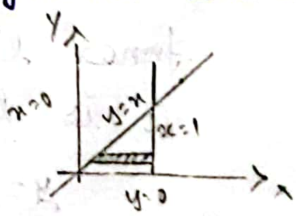
y limit $y=0$ to $y=x$

Inner limit is with respect to y.

∴ It is verticle strip



By changing order of integration we have to draw a horizontal strip



x limit $x=y$ to $x=1$

y limit $y=0$ to $y=1$

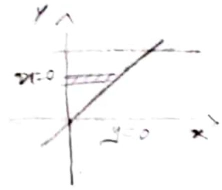
$$I = \int_0^1\int_y^1 dx\,dy$$

2. Change the order of integration for $\int_0^1\int_0^y dx\,dy$

Sol

Given integral is in the correct form

$$\int_0^1\int_0^y dx\,dy$$



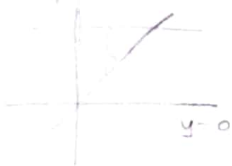
Given limits : x limit $x=0$ to $x=y$

y limit $y=0$ to $y=1$

Inner limit is with respect to x.

$\therefore$ It is horizontal strip.

By changing order of integration we have to draw vertical strip



x limit : $x=0$ to $x=1$

y limit : $y=x$ to $y=1$

$$I=\int_0^1\int_x^1 dy\,dx$$

3. Evaluate by changing the order of integration $\int_0^4\int_{x^2/4}^{2\sqrt{x}} dy\,dx$

Sol

Given integral is in the correct form

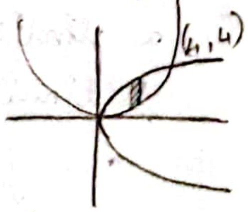
$$I=\int_0^4\int_{x^2/4}^{2\sqrt{x}} dy\,dx$$

Given limits: x limit $x=0$ to $x=4$

y limit $y=x^2/4$ to $2\sqrt{x}$

Inner limit is with respect to y.

It is the vertical strip



$y=\frac{x^2}{4}$, $y=2\sqrt{x}$

$4y=x^2 \rightarrow (1)$ $y^2=4x \rightarrow (2)$

$x^2=4y$ squaring on both side

$x^4=16y^2$

$x^4 = 16(4x)$

$x^4 = 64x$

$x^3 = 64$

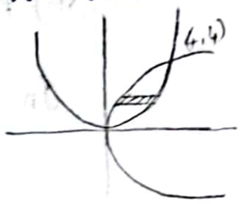
$\boxed{x=4}$ sub in ②

$y^2 = 4x$

$y^2 = 4(4) = 16$

$\boxed{y=4}$

By changing order of integration we have to draw horizontal strip



x limit $x = \frac{y^2}{4}$ to $x = 2\sqrt{y}$

y limit $y=0$ to $y=4$

$$I = \int_0^4 \int_{y^2/4}^{2\sqrt{y}} dx\,dy$$

$$= \int_0^4 [x]_{y^2/4}^{2\sqrt{y}}\,dy$$

$$= \int_0^4 \left[2\sqrt{y} - \frac{y^2}{4}\right] dy$$

$$= \int_0^4 \left(2y^{1/2} - \frac{y^2}{4}\right) dy$$

$$I = \left[\frac{2y^{3/2}}{3/2} - \frac{y^3}{4\times 3}\right]_0^4$$

$$= \frac{2(4)^{3/2}}{3/2} - \frac{4^3}{12}$$

$$= \frac{2\times 4\sqrt{4}}{3/2} - \frac{4^2}{3}$$

$$= \frac{32-16}{3} = \frac{16}{3}$$

4. Evaluate $\int_0^1 \int_{x^2}^{2-x} xy\,dy\,dx$ using change of order of integration.

Sol

The given integral is in correct form

$$I = \int_0^1 \int_{x^2}^{2-x} xy\,dy\,dx$$

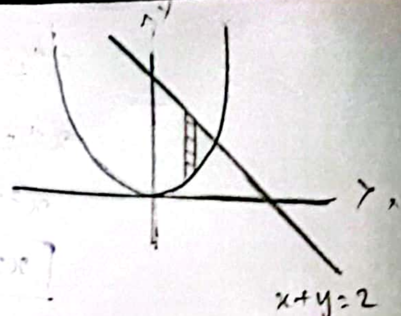
Given limit x limit : $x=0$ to $x=1$

y limit : $y = x^2$ to $y = 2-x$

Inner limit is with respect to y

$\therefore$ It is vertical strip

By changing the order of integration we have to draw a horizontal strip

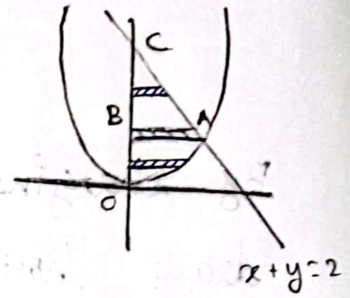


As there is parabola and straight line we have to draw two horizontal strip

There are two regions

i) OAB

ii) ABC . $I = I_1 + I_2$



In the region OAB

x limit $x=0$ to $x=\sqrt{y}$

y limit $y=0$ to $y=1$

$I_1 = \int_0^1 \int_0^{\sqrt{y}} xy \, dy \, dx$. It is not in correct of integration

$$I_1 = \int_0^1 \int_0^{\sqrt{y}} xy \, dx \, dy = \int_0^1 \left[\frac{x^2}{2} y \right]_0^{\sqrt{y}} dy$$

$$= \int_0^1 \left(\frac{(\sqrt{y})^2 y}{2} \right) dy$$

$$= \int_0^1 y^2/2 \, dy$$

$$= \left(\frac{y^3}{6} \right)_0^1 = 1/6$$

In the region ABC

x limit : $x=0$ to $x=2-y$

y limit : $y=1$ to $y=2$

$$I_2 = \int_1^2 \int_0^{2-y} xy \, dy \, dx$$

It is not correct form of integration

$$I = \int_1^2 \int_0^{2-y} xy \, dx \, dy$$

$$= \int_1^2 \left[\frac{x^2}{2} y\right]_0^{2-y} y\,dy.$$

$$= \int_1^2 \left[\frac{x^2}{2}\right]_0^{2-y} y\,dy$$

$$= \int_1^2 \left(\frac{(2-y)^2}{2}\right) y\,dy$$

$$= \int_1^2 \left(\frac{4+y^2-4y}{2}\right) y\,dy$$

$$= \frac{1}{2}\int_1^2 (4y + y^3 - 4y^2)\,dy$$

$$= \frac{1}{2}\left[\frac{\cancel{4}^{2} y^2}{\cancel{2}} + \frac{y^4}{4} - \frac{4y^3}{3}\right]_1^2$$

$$= \frac{1}{2}\left[\left(2\times 4 + 4 - \frac{32}{3}\right) - \left(2 + \frac{1}{4} - \frac{4}{3}\right)\right]$$

$$= \frac{1}{2}\left[\frac{(12\times 3)-32}{3} - \frac{24+3-16}{12}\right]$$

$$= \frac{1}{2}\left(\frac{4}{3} - \frac{11}{12}\right) = \frac{1}{2}\left(\frac{16-11}{12}\right)$$

$$= \frac{5}{24}$$

$$I = I_1 + I_2$$

$$= \frac{1}{6} + \frac{5}{24} = \frac{4+5}{24} = 9/24$$

$$I = 3/8$$

5. Evaluate $\int_0^a \int_{x^2/a}^{2a-x} xy\,dy\,dx$ using change of order of integration.

Sol:

Given integration is in the correct form

$$I = \int_0^a \int_{x^2/a}^{2a-x} xy\,dy\,dx$$

Given limit x limit $x=0$ to $x=a$

y limit $y = x^2/a$ to $y = 2a-x$

ie, $y = x^2/a$ to $y = 2a - x$

$x^2 = ay$ $x + y = 2a$

Inner limit with respect to y.

∴ it is vertical strip.

By changing order of integration

As there is parabola & straight line we have to draw two horizontal strip

There are two region

i) OAB

ii) ABC $I = I_1 + I_2$

In the region OAB

x limit $x = 0$ to $x = \sqrt{ay}$

y limit $y = 0$ to $y = a$

$$I_1 = \iint xy\, dy\, dx$$

I_1 is not in the correct form of integration.

$$I_1 = \int_0^a \int_0^{\sqrt{ay}} xy\, dx\, dy$$

$$= \int_0^a \left[\frac{x^2}{2}\right]_0^{\sqrt{ay}} y\, dy$$

$$= \int_0^a \left(\frac{ay}{2}\right) y\, dy = \int_0^a \frac{ay^2}{2}\, dy$$

$$= \frac{a}{2}\left[\frac{y^3}{3}\right]_0^a = \frac{a^4}{6}$$

In the region ABC

x limit : $x = 0$ to $x = 2a - y$

y limit : $y = a$ to $y = 2a$

$$I_2 = \int_a^{2a}\int_0^{2a-y} xy\, dx\, dy$$

$$= \int_a^{2a} \left[\frac{x^2}{2}\cdot y\right]_0^{2a-y} dy$$

$$= \int_a^{2a} \left[\frac{(2a-y)^2}{2} y\right] dy$$

$$= \frac{1}{2}\int_a^{2a} 4a^2y + y^3 - 4ay^2\, dy$$

$$= \frac{1}{2}\left[\frac{4a^2y^2}{2} + \frac{y^4}{4} - \frac{4ay^3}{3}\right]_a^{2a}$$

$$= \frac{1}{2}\left[\frac{16a^4}{2} + \frac{16a^4}{4} - \frac{32a^4}{3}\right] - \left[\frac{4a^4}{2} + \frac{a^4}{4} - \frac{4a^4}{3}\right]$$

$$= \frac{1}{2}\left[\frac{12a^4}{2} + \frac{15a^4}{4} - \frac{28a^4}{3}\right]$$

$$= \frac{1}{2}\left(\frac{72a^4 + 45a^4 - 112a^4}{12}\right)$$

$$= \frac{1}{24} \times 5a^4$$

$$I = I_1 + I_2$$

$$= \frac{a^4}{6} + \frac{5a^4}{24}$$

$$= \frac{4a^4 + 5a^4}{24} = \frac{9a^4}{24}$$

$$= \frac{3a^4}{8}$$