Construction of Analytic function:

Milne-Thomson method:

Let $f(z) = u + iv$ is to be constructed.

(i) Suppose the real part u is given, then

$$f(z) = \int [\varphi_1(z,0) - i\varphi_2(z,0)]dz$$

where $\varphi_1(z,0) = \frac{\partial u}{\partial x}(z,0)$, $\varphi_2(z,0) = \frac{\partial u}{\partial y}(z,0)$.

(ii) Suppose imaginary part v is given, then

$$f(z) = \int [\varphi_1(z,0) + i\varphi_2(z,0)]dz$$

where $\varphi_1(z,0) = \frac{\partial v}{\partial y}(z,0)$, $\varphi_2(z,0) = \frac{\partial v}{\partial x}(z,0)$

(1) Show that the fn $u = x^3 + x^2 - 3xy^2 + 2xy - y^2$ is harmonic and find the corresponding analytic fn $f(x) = u + iv$.

Soln:

$$u = x^3 + x^2 - 3xy^2 + 2xy - y^2$$

$$u_x = 3x^2 + 2x - 3y^2 + 2y \; ; \; u_y = -6xy + 2x - 2y$$

$$u_{xx} = 6x + 2 \; ; \; u_{yy} = -6x - 2$$

$$u_{xx} + u_{yy} = 0$$

$\Rightarrow$ u is harmonic. $\quad f(z) = \int \left[\frac{\partial u}{\partial x}(z,0) - i \frac{\partial u}{\partial y}(z,0) \right] dz$

$$f(z) = \int [\phi_1(z,0) - i\, \phi_2(z,0)]\, dz$$

where $\phi_1(z,0) = \frac{\partial u}{\partial x}(z,0) = 3z^2 + 2z$

$$\phi_2(z,0) = \frac{\partial u}{\partial y}(z,0) = 2z$$

$$f(z) = \int [(3z^2 + 2z) - i \cdot 2z]\, dz$$

$$= \frac{3z^3}{3} + \frac{2z^2}{2} - i \cdot \frac{2z^2}{2} + c$$

$$f(z) = z^3 + z^2(1 - i) + c.$$

(2) Find an analytic fn whose imaginary part is $v = e^{2x}(y \cos 2y + x \sin 2y)$.

Soln:

$$v = e^{2x}(y \cos 2y + x \sin 2y)$$

$$v_x = 2e^{2x}(y \cos 2y + x \sin 2y) + e^{2x} \sin 2y$$

$$v_y = e^{2x}(\cos 2y + y(-2\sin 2y) + 2x \cos 2y)$$

$$f(z) = \int [\phi_1(z,0) + i\, \phi_2(z,0)]\, dz$$

$$= \int \left[\frac{\partial v}{\partial y}(z,0) + i \frac{\partial v}{\partial x}(z,0) \right] dz$$

$$\phi_1(z,0) = \frac{\partial v}{\partial y}(z,0) = e^{2z}(1+0+2z)$$
$$= e^{2z} + 2ze^{2z}$$
$$\phi_2(z,0) = \frac{\partial v}{\partial x}(z,0) = 2e^{2z}(0) + e^{2z}(0) = 0.$$
$$f(z) = \int [e^{2z} + 2ze^{2z}]\,dz + c$$
$$= \int e^{2z}dz + 2\int ze^{2z}dz + c$$
$$= \frac{e^{2z}}{2} + 2\left(\frac{ze^{2z}}{2} - \frac{e^{2z}}{4}\right) + c$$
$$= \frac{e^{2z}}{2} + ze^{2z} - \frac{e^{2z}}{2} + c$$
$$= ze^{2z} + c.$$

(3) Given that $u = \dfrac{\sin 2x}{\cosh 2y - \cos 2x}$, find the

(*) analytic f^n whose real part is u.

Soln:

$$u = \frac{\sin 2x}{\cosh 2y - \cos 2x}$$
$$u_x = \frac{(\cosh 2y - \cos 2x)\cos 2x.2 - \sin 2x(2\sin 2x)}{(\cosh 2y - \cos 2x)^2}$$
$$u_y = \frac{(\cosh 2y - \cos 2x)(0) - \sin 2x(2\sinh 2y)}{(\cosh 2y - \cos 2x)^2}$$
$$f(z) = \int [\phi_1(z,0) - i\phi_2(z,0)]\,dz$$
$$\phi_1(z,0) = \frac{\partial u}{\partial x}(z,0)$$
$$= \frac{(\cos 0 - \cos 2z)2\cos 2z - \sin 2z(2\sin 2z)}{(\cos 0 - \cos 2z)^2}$$

$$= \frac{2\cos 2z - (2\cos^2 2z - 2\sin^2 2z)}{(1-\cos 2z)^2}$$

$$= \frac{2\cos 2z - 2}{(1-\cos 2z)^2}$$

$$= \frac{-2(1-\cos 2z)}{(1-\cos 2z)^2}$$

$$= \frac{-2}{1-\cos 2z} = \frac{-2}{2\sin^2 z} = -\operatorname{cosec}^2 z$$

$$\phi_2(z,0) = \frac{(\cos 0 - \cos 2z)(0) - \sin 2z(2\sin 0)}{(\cos 0 - \cos 2z)^2}$$

$$= 0.$$

$$f(z) = \int -\operatorname{cosec}^2 z \, dz = \cot z + c$$

$$f(z) = \cot z + c$$