UNIT - III

COMPLEX DIFFERENTIATION

INTRODUCTION:

If x and y are real numbers then $z = x + iy$ is called a complex number where x is called real part of z, y is called the imaginary part of z and the value of i is $\sqrt{-1}$. The complex number $x - iy$ is called as the complex conjugate of z & it is denoted by $\bar{z}$.

i.e., $\bar{z} = x - iy$.

NOTE :

1. $|z| = \sqrt{x^2 + y^2}$
2. $|z^2| = z\bar{z}$
3. $z\bar{z} = x^2 + y^2 = r^2$
4. $|\bar{z}| = |z|$
5. Real part of $z = \dfrac{z + \bar{z}}{2}$
6. Imaginary part of $z = \dfrac{z - \bar{z}}{2i}$
7. $z = re^{i\theta}$ is called polar form of z.
8. Amplitude of $z = \theta = \tan^{-1}(y/x)$

FUNCTIONS OF COMPLEX VARIABLE :

$w = f(z) = u(x, y) + iv(x, y)$ where $u(x, y)$ and $v(x, y)$ are real variables.

Single Valued function:

If for each value of z in R there will be only one value of w, then w is called a single valued function of z.

Ex: $w = z^2$; $w = 1/z$.

$w = 1/z$

z : 1	2	-2	3		z : 1	2	-2	3
w : 1	4	4	9		w : 1	$1/2$	$-1/2$	$1/3$

Multiple-Valued function:

If there is more than one value of w corresponding to a given value of z, then w is called a multiple-valued function.

Ex: $w = z^{1/2}$

z : 4   9   1

w : $-2, 2$ $-3, 3$ $1, -1$

Analytic function:

A function $f(z)$ is said to be analytic at a point $z = a$ in a region R if

(i) $f(z)$ is differentiable at $z = a$.

(ii) $f(z)$ is differentiable at all points for some neighbourhood of $z = a$.

(or)

A function is said to be analytic at a point if its derivative exists not only at that point but also in some neighbourhood of that point.

Necessary Condition (Cartesian Coordinates):
(or) Cauchy-Riemann equations:

If the function $f(z) = u(x,y) + iv(x,y)$ is analytic in a region R of the z plane, then

(i) $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ exists

(ii) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

at every point in that region.

Sufficient Conditions:

If the function $f(z) = u(x,y) + iv(x,y)$ is analytic in a region R of the z-plane if

(i) u_x, u_y, v_x & v_y are exists and all are Continuous.

(ii) $u_x = v_y$ and $u_y = -v_x$.

Necessary Condition (polar Coordinates):

If the function $w = f(z) = u(r,\theta) + iv(r,\theta)$ is analytic in a region R of the z-plane then

(i) if $\frac{\partial u}{\partial r}, \frac{\partial u}{\partial \theta}, \frac{\partial v}{\partial r}$ and $\frac{\partial v}{\partial \theta}$ exists.

(ii) $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$ and $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$

Sufficient Conditions:

If the function $w = f(z) = u(r,\theta) + iv(r,\theta)$ is analytic in a region R of the z-plane, then

(i) $\frac{\partial u}{\partial r}, \frac{\partial u}{\partial \theta}, \frac{\partial v}{\partial r}$ and $\frac{\partial v}{\partial \theta}$ exists & all are Continuous

(ii) $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$ and $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$

PROBLEMS:

(1) Prove that $w = z^2$ is analytic.

Soln:

We know $z = x + iy$

$$\therefore \quad w = z^2$$
$$= (x+iy)^2$$
$$= x^2 - y^2 + 2ixy$$

$$u + iv = (x^2 - y^2) + i(2xy)$$

$u = x^2 - y^2$; $v = 2xy$

$u_x = 2x$ $v_x = 2y$

$u_y = -2y$ $v_y = 2x$

$$u_x = v_y \ \& \ u_y = -v_x$$

It satisfies CR equations.

$\Rightarrow w = z^2$ is analytic.

(2) Determine whether the function $w = 2xy + i(x^2 - y^2)$ is analytic.

Soln: $w = 2xy + i(x^2 - y^2)$

$$u + iv = 2xy + i(x^2 - y^2)$$

$u = 2xy$, $v = x^2 - y^2$

$u_x = 2y$, $v_x = 2x$

$u_y = 2x$ $v_y = -2y$

$$u_x \neq v_y$$
$$\& \ u_y \neq -v_x$$

It doesn't satisfies CR equations

$\Rightarrow w = 2xy + i(x^2 - y^2)$ is not analytic.

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③ Verify whether $f(z) = \sinh z$ is analytic using CR eqns.

Soln:

$\sin i\theta = i \sinh \theta$

$\cos i\theta = \cosh \theta$

$f(z) = \sinh z$

$u + iv = \sinh(x+iy)$

$= \frac{1}{i} \sin i(x+iy)$ (multiply & divide by i)

$= \frac{1}{i} \sin(ix + i^2 y)$

$= \frac{1}{i} \sin(ix - y)$

$= \frac{1}{i} [\sin ix \cos y - \cos ix \sin y]$

$= \frac{1}{i} [i \sinh x \cos y - \cosh x \sin y]$

$= \sinh x \cos y - \frac{1}{i} \cosh x \sin y$

$= \sinh x \cos y + i \cosh x \sin y \qquad \left[\frac{1}{i} = -i\right]$

$u = \sinh x \cos y \quad , \quad v = \cosh x \sin y$

$u_x = \cosh x \cos y \quad , \quad v_x = \sinh x \sin y$

$u_y = -\sinh x \sin y \quad , \quad v_y = \cosh x \cos y$

$u_x = v_y$ & $u_y = -v_x$

$\therefore$ It satisfies CR equations.

$\therefore f(z) = \sinh z$ is analytic.

④ Show that $f(z) = |z|^2$ is nowhere analytic.

Soln:

$f(z) = |z|^2$

$u + iv = x^2 + y^2$

$u = x^2 + y^2 \quad , \quad v = 0$

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$u_x = 2x$, $v_x = 0$

$u_y = 2y$, $v_y = 0$

$\Rightarrow u_x \neq v_y$ & $u_y \neq -v_x$

It doesn't satisfies CR eqns.

$\therefore f(z) = |z|^2$ is nowhere analytic.

(5) If $u+iv$ is analytic then $v-iu$ is also analytic.

Soln:

$u+iv$ is analytic

i.e., CR eqns are satisfied.

i.e., $u_x = v_y$ & $u_y = -v_x$.

i.e., $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

To prove: $v-iu$ is also analytic.

i.e., We have to prove, $\frac{\partial v}{\partial x} = \frac{\partial(-u)}{\partial y}$ & $\frac{\partial v}{\partial y} = -\frac{\partial(-u)}{\partial x}$

$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$ & $\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}$

We know that,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\Rightarrow \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \text{ and } \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}$$

Hence $v-iu$ is also analytic.

(6) If $w = e^z$, find $\frac{dw}{dz}$ using complex variable.

Soln:

$w = e^z$.

$u+iv = e^{x+iy} = e^x \cdot e^{iy} = e^x(\cos y + i\sin y)$

$= e^x \cos y + i\, e^x \sin y$

$u = e^x \cos y$, $v = e^x \sin y$

$u_x = e^x \cos y$, $v_x = e^x \sin y$.

[Result : If $w = f(z) = u+iv$ then

$$\frac{dw}{dz} = f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

finding $\frac{dw}{dz}$ in terms of partial derivatives w.r.t x)

$$\Rightarrow \frac{dw}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= e^x \cos y + i e^x \sin y$$

$$= e^x (\cos y + i \sin y)$$

$$= e^x e^{iy}$$

$$= e^{x+iy}$$

$$= e^z .$$