Green's Theorem in a Plane:

If R is a closed region of the XY Plane bounded by a simple closed curve C and if M and N are continuous functions of x and y having continuous derivatives in R then

$$\boxed{\int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy}$$

where C is a curve traversed in the anticlockwise direction.

Problems:

1. Evaluate by Green's theorem $\int_C (xy+x^2)\,dx + (x^2+y^2)\,dy$ where C is the square formed by $x=-1$, $x=1$, $y=-1$, $y=1$.

Solution:

Let R be the region enclosed by C.

By Green's theorem,

$$\int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

Here $M = xy + x^2 \Rightarrow \frac{\partial M}{\partial y} = x$

$N = x^2 + y^2 \Rightarrow \frac{\partial N}{\partial x} = 2x$

$$\int_C (xy + x^2)\,dx + (x^2+y^2)\,dy = \iint_R (2x - x)\,dx\,dy.$$

$$= \int_{-1}^{1}\int_{-1}^{1} x\,dx\,dy$$

$$= \int_{-1}^{1} \left[\frac{x^2}{2}\right]_{-1}^{1} dy = 0$$

(2) Evaluate by Green's theorem $\int_C e^{-x}(\sin y\,dx + \cos y\,dy)$ where C is the rectangle with vertices $(0,0)$, $(\pi,0)$, $(\pi,\pi/2)$, $(0,\pi/2)$.

Soln:

Let R be the region enclosed by C.

By Green's theorem,

$$\int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

Here $M = e^{-x}\sin y \Rightarrow \frac{\partial M}{\partial y} = e^{-x}\cos y$

$N = e^{-x}\cos y \Rightarrow \frac{\partial N}{\partial x} = -e^{-x}\cos y$

$$\therefore \int_C e^{-x}(\sin y\,dx + \cos y\,dy) = \iint_R (-e^{-x}\cos y - e^{-x}\cos y)\,dx\,dy$$

$$= \int_0^{\pi/2} \int_0^{\pi} (-2e^{-x} \cos y)\, dx\, dy$$

$$= -2 \int_0^{\pi/2} \int_0^{\pi} e^{-x} \cos y \, dx\, dy$$

$$= 2(e^{-\pi} - 1).$$

③ Evaluate by Green's theorem

$\int_C (x^2 - \cosh y)\,dx + (y + \sin x)\,dx$ where c is the rectangle with vertices $(0,0)$, $(\pi, 0)$, $(\pi, 1)$, $(0,1)$.

Soln:

$\pi(\cosh 1 - 1)$.

④ Verify Green's theorem in the plane for

$\int_C (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy$ where c is the boundary of the region defined by $x = y^2$, $y = x^2$.

Soln:

By Green's theorem,

$$\int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\, dy$$

Given: $\int_C (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy$

Here $M = 3x^2 - 8y^2$, $N = 4y - 6xy$

$$\frac{\partial M}{\partial y} = -16y \quad , \quad \frac{\partial N}{\partial x} = -6y$$

Step 1:

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\, dy = \iint_R (-6y + 16y)\, dx\, dy$$

$$= \int_0^1 \int_{y^2}^{\sqrt{y}} 10y \, dx \, dy$$

$$= \int_0^1 10y \, [x]_{y^2}^{\sqrt{y}} \, dy = \int_0^1 10y \, (\sqrt{y} - y^2) \, dy$$

$$= 10 \int_0^1 (y^{3/2} - y^3) \, dy = 10 \left[\frac{y^{5/2}}{5/2} - \frac{y^4}{4} \right]_0^1$$

$$= 10 \left(\frac{2}{5} - \frac{1}{4} \right) = 10 \left(\frac{8-5}{20} \right) = 3/2 \longrightarrow (1)$$

Step 2 :

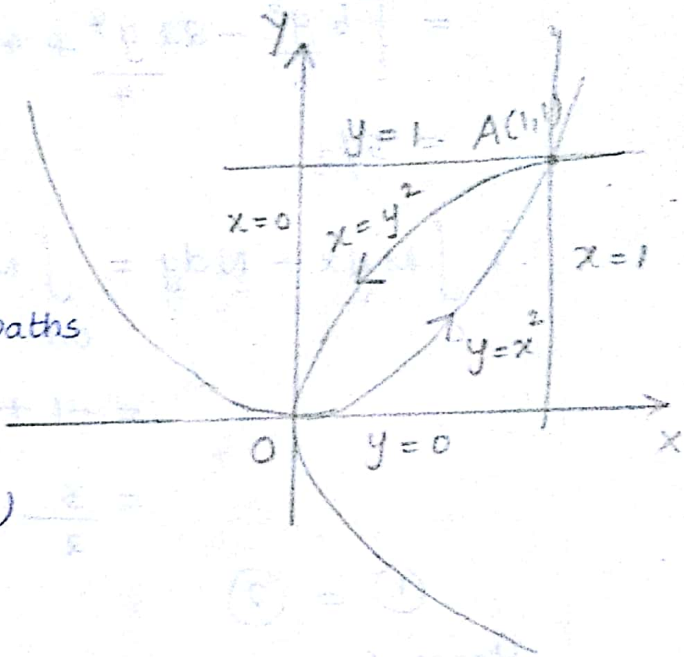
To evaluate

$\int_C M dx + N dy$ we

take C in different paths

(i) along OA $(y = x^2)$

(ii) along AO $(x = y^2)$



(i) Along OA :

$$\int_{OA} M dx + N dy = \int_{OA} [3x^2 - 8x^4] \, dx + [4x^2 - 6x \cdot x^2] \, 2x \, dx$$

$(\because x^2 = y, \ 2x dx = dy)$

$$= \int_0^1 (3x^2 - 8x^4 + 8x^3 - 12x^4) \, dx$$

$(\because$ Along OA, x varies from 0 to 1)

$$= \int_0^1 (-20x^4 + 8x^3 + 3x^2) \, dx$$

$$= \left[-20\frac{x^5}{5} + \frac{8x^4}{4} + \frac{3x^3}{3} \right]_0^1 = -1$$

(ii) Along AO :

$$\int_{AO} M\,dx + N\,dy = \int_{AO} (3y^4 - 8y^2)\, 2y\,dy + (4y - 6y y^2)\,dy$$

$(\because y^2 = x,\ 2y\,dy = dx)$

$$= \int_{AO} (6y^5 - 16y^3 + 4y - 6y^3)\,dy$$

$$= \int_1^0 (6y^5 - 22y^3 + 4y)\,dy \quad (\because \text{Along AO } y \text{ varies from 1 to 0}).$$

$$= \left[6\frac{y^6}{6} - 22\frac{y^4}{4} + 4\frac{y^2}{2} \right]_1^0$$

$= 5/2$.

$$\therefore \int_C M\,dx + N\,dy = \int_{OA} M\,dx + N\,dy + \int_{AO} M\,dx + N\,dy$$

$$= -1 + \frac{5}{2}$$

$$= \frac{3}{2} \longrightarrow (2)$$

(1) = (2)

Hence Green's theorem is verified.