Unit Normal :

A unit normal to the given surface φ at the point is $\dfrac{\nabla\varphi}{|\nabla\varphi|}$

Directional Derivative :

The directional derivative of φ in the direction $\vec{a}$ is given by,

$$\nabla\varphi \cdot \frac{\vec{a}}{|\vec{a}|} \quad (\text{or}) \quad \nabla\varphi \cdot \hat{n} \quad \text{where } \hat{n} = \frac{\vec{a}}{|\vec{a}|}$$

The directional derivative is maximum in the direction of the normal to the given surface. Its maximum value is $|\nabla\varphi|$.

Angle between two surfaces :

$$\boxed{\cos\theta = \frac{\nabla\varphi_1 \cdot \nabla\varphi_2}{|\nabla\varphi_1|\,|\nabla\varphi_2|}}$$

Note :

If the surfaces cut orthogonally then,

$$\boxed{\nabla\varphi_1 \cdot \nabla\varphi_2 = 0}$$

Scanned by CamScanner

Problems :

① Find a unit normal to the surface $x^2y + 2xz = 4$ at $(2,-2,3)$

Soln :

$$\varphi : x^2y + 2xz - 4$$

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$= \vec{i}\frac{\partial}{\partial x}(x^2y + 2xz - 4) + \vec{j}\frac{\partial}{\partial y}(x^2y + 2xz - 4) + \vec{k}\frac{\partial}{\partial z}(x^2y + 2xz - 4)$$

$$= \vec{i}(2xy + 2z) + \vec{j}(x^2) + \vec{k}(2x)$$

$$\nabla\varphi_{(2,-2,3)} = \vec{i}(-8+6) + \vec{j}(4) + \vec{k}(4)$$

$$= -2\vec{i} + 4\vec{j} + 4\vec{k}$$

$$|\nabla\varphi| = \sqrt{4+16+16} = \sqrt{36} = 6$$

Unit normal to the given surface at $(2,-2,3)$

$$= \frac{\nabla\varphi}{|\nabla\varphi|} = \frac{-2\vec{i} + 4\vec{j} + 4\vec{k}}{6}$$

$$= \frac{1}{3}(-\vec{i} + 2\vec{j} + 2\vec{k})$$

② Find the unit vector normal to $x^2 - y^2 + z = 2$ at $(1,-1,2)$.

Soln :

$$\frac{\nabla\varphi}{|\nabla\varphi|} = \frac{2\vec{i} + 2\vec{j} + \vec{k}}{3}$$

③ Find the unit vector normal to $x^2 + xy + z^2 = 4$ at $(1,-1,2)$

Soln:

$$\frac{\nabla\varphi}{|\nabla\varphi|} = \frac{\vec{i} + \vec{j} + 4\vec{k}}{\sqrt{18}}$$

(4) Find the directional derivative of the function $x^2 + 2xy$ at $(1,-1,3)$ in the direction $\vec{i} + 2\vec{j} + 2\vec{k}$

Soln:

$$\varphi = x^2 + 2xy$$

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$= \vec{i}\frac{\partial}{\partial x}(x^2+2xy) + \vec{j}\frac{\partial}{\partial y}(x^2+2xy) + \vec{k}\frac{\partial}{\partial z}(x^2+2xy)$$

$$= \vec{i}(2x+2y) + \vec{j}(2x) + \vec{k}(0)$$

$$\nabla\varphi_{(1,-1,3)} = \vec{i}(2-2) + \vec{j}(2) = 2\vec{j}$$

Given: $\vec{a} = \vec{i} + 2\vec{j} + 2\vec{k}$

$$|\vec{a}| = \sqrt{1+4+4} = 3$$

$$\hat{n} = \frac{\vec{a}}{|\vec{a}|} = \frac{\vec{i}+2\vec{j}+2\vec{k}}{3}$$

$$\nabla\varphi \cdot \hat{n} = 2\vec{j} \cdot \left[\frac{\vec{i}+2\vec{j}+2\vec{k}}{3}\right] = \frac{4}{3}$$

$$\boxed{\nabla\varphi \cdot \hat{n} = \frac{4}{3}}$$

(5) Find the directional derivative of $xy + yz + zx$ at $(1,1,1)$ in the direction $\vec{i} + \vec{j}$.

Soln:

$2\sqrt{2}$

(6) Find the directional derivative of $3x^2 + 2y - 3z$ at $(1,1,1)$ in the direction $2\vec{i} + 2\vec{j} - \vec{k}$.

Soln:

$\frac{19}{3}$

Scanned by CamScanner

(7) What is the greatest rate of increase of $\varphi = xyz^2$ at $(1,0,3)$?

Soln: Let $\varphi = xyz^2$

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$= \vec{i}\frac{\partial}{\partial x}(xyz^2) + \vec{j}\frac{\partial}{\partial y}(xyz^2) + \vec{k}\frac{\partial}{\partial z}(xyz^2)$$

$$= \vec{i}(yz^2) + \vec{j}(xz^2) + \vec{k}(2xyz)$$

$$\nabla\varphi_{(1,0,3)} = \vec{i}(0) + \vec{j}(9) + \vec{k}(0)$$

$$= 9\vec{j}$$

Maximum (or) Greatest rate of increase $= |\nabla\varphi|$

$$= \sqrt{9^2}$$

$$= 9$$

(8) In what direction from the point $(1,-1,2)$ is the directional derivative of $\varphi = x^2y^2z^3$ a maximum? What is the magnitude of this maximum?

Soln: $\varphi = x^2y^2z^2$

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$= \vec{i}\frac{\partial}{\partial x}(x^2y^2z^2) + \vec{j}\frac{\partial}{\partial y}(x^2y^2z^2) + \vec{k}\frac{\partial}{\partial z}(x^2y^2z^2)$$

$$= 2xy^2z^2\,\vec{i} + 2x^2yz^2\,\vec{j} + 2x^2y^2z\,\vec{k}$$

$\nabla\varphi_{(1,-1,2)} = 16\vec{i} - 16\vec{j} + 12\vec{k}$ is the directional derivative.

Magnitude is $|\nabla\varphi| = \sqrt{16^2+16^2+12^2} = \sqrt{656}$

(9) Find the directional derivative of $\varphi = xy^2z^3$ at the point $(1,1,1)$ along the normal to the surface $x^2 + xy + z^2 = 3$ at the point $(1,1,1)$.

Soln: $\nabla\varphi$ is normal to the surface x^2+xy+z^2-3

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$= \vec{i}\frac{\partial}{\partial x}(x^2+xy+z^2-3) + \vec{j}\frac{\partial}{\partial y}(x^2+xy+z^2-3) + \vec{k}\frac{\partial}{\partial z}(x^2+xy+z^2-3)$$

$$= \vec{i}(2x+y) + \vec{j}(x) + \vec{k}(2z)$$

$$\nabla\varphi_{(1,1,1)} = 3\vec{i}+\vec{j}+2\vec{k}$$

To find the directional derivative of $\varphi = xy^2z^3$ at $(1,1,1)$ in the direction $\vec{a} = 3\vec{i}+\vec{j}+2\vec{k}$.

$$\nabla\varphi = \vec{i}\frac{\partial}{\partial x}(xy^2z^3) + \vec{j}\frac{\partial}{\partial y}(xy^2z^3) + \vec{k}\frac{\partial}{\partial z}(xy^2z^3)$$

$$= \vec{i}(y^2z^3) + \vec{j}(2xyz^3) + \vec{k}(3xy^2z^2)$$

$$\nabla\varphi_{(1,1,1)} = \vec{i}+2\vec{j}+3\vec{k}$$

$$\text{Directional derivative} = \nabla\varphi\cdot\frac{\vec{a}}{|\vec{a}|}$$

$$= \frac{(\vec{i}+2\vec{j}+3\vec{k})\cdot(3\vec{i}+\vec{j}+2\vec{k})}{\sqrt{9+1+4}}$$

$$= \frac{3+2+6}{\sqrt{14}}$$

$$= \frac{11}{\sqrt{14}}$$

Scanned by CamScanner

(10) Find the angle between the surfaces $x^2+y^2+z^2=5$ and $x^2+y^2+z^2-2x=5$ at $(0,1,2)$

Soln :

Let $\varphi_1 : x^2+y^2+z^2-5$; $\varphi_2 = x^2+y^2+z^2-2x-5$

$$\frac{\partial\varphi_1}{\partial x} = 2x \qquad \frac{\partial\varphi_2}{\partial x} = 2x-2$$

$$\frac{\partial\varphi_1}{\partial y} = 2y \qquad \frac{\partial\varphi_2}{\partial y} = 2y$$

$$\frac{\partial\varphi_1}{\partial z} = 2z \qquad \frac{\partial\varphi_2}{\partial z} = 2z$$

$$\nabla\varphi_1 = 2x\vec{i}+2y\vec{j}+2z\vec{k} \; ; \; \nabla\varphi_2 = (2x-2)\vec{i}+2y\vec{j}+2z\vec{k}$$

$$\nabla\varphi_{1(0,1,2)} = 2\vec{j}+4\vec{k} \qquad \nabla\varphi_{2(0,1,2)} = -2\vec{i}+2\vec{j}+4\vec{k}$$

$$|\nabla\varphi_1| = \sqrt{4+16} = \sqrt{20} \qquad |\nabla\varphi_2| = \sqrt{4+4+16} = \sqrt{24}$$

Angle between the surfaces,

$$\cos\theta = \frac{\nabla\varphi_1 \cdot \nabla\varphi_2}{|\nabla\varphi_1|\,|\nabla\varphi_2|}$$

$$= \frac{(2\vec{j}+4\vec{k})\cdot(-2\vec{i}+2\vec{j}+4\vec{k})}{\sqrt{20}\sqrt{24}}$$

$$= \frac{4+16}{\sqrt{20}\sqrt{24}} = \frac{20}{\sqrt{20}\sqrt{24}}$$

$$\cos\theta = \sqrt{\frac{5}{6}}$$

$$\boxed{\theta = \cos^{-1}\sqrt{\frac{5}{6}}}$$

(11) Find the angle between the surfaces $x \log z = y^2 - 1$ and $x^2 y = 2 - z$ at the point $(1,1,1)$.

Soln:

$\varphi_1 : x \log z - y^2 + 1$ $\qquad$ $\varphi_2 : x^2 y - 2 + z$

$\frac{\partial \varphi_1}{\partial x} = \log z$ $\qquad$ $\frac{\partial \varphi_2}{\partial x} = 2xy$

$\frac{\partial \varphi_1}{\partial y} = -2y$ $\qquad$ $\frac{\partial \varphi_2}{\partial y} = x^2$

$\frac{\partial \varphi_1}{\partial z} = \frac{x}{z}$ $\qquad$ $\frac{\partial \varphi_2}{\partial z} = 1$

$\nabla \varphi_1 = \log z \, \vec{i} - 2y\vec{j} + \frac{x}{z}\vec{k}$ $\qquad$ $\nabla \varphi_2 = (2xy)\vec{i} + x^2\vec{j} + \vec{k}$

$\nabla \varphi_{1(1,1,1)} = -2\vec{j} + \vec{k}$ $\qquad$ $\nabla \varphi_{2(1,1,1)} = 2\vec{i} + \vec{j} + \vec{k}$

$|\nabla \varphi_1| = \sqrt{4+1} = \sqrt{5}$ $\qquad$ $|\nabla \varphi_2| = \sqrt{4+1+1} = \sqrt{6}$

$$\cos\theta = \frac{\nabla \varphi_1 \cdot \nabla \varphi_2}{|\nabla \varphi_1| \, |\nabla \varphi_2|}$$

$$= \frac{(-2\vec{j} + \vec{k}) \cdot (2\vec{i} + \vec{j} + \vec{k})}{\sqrt{5} \cdot \sqrt{6}}$$

$$= \frac{-2+1}{\sqrt{30}} = \frac{-1}{\sqrt{30}}$$

$$\boxed{\theta = \cos^{-1}\left(\frac{-1}{\sqrt{30}}\right)}$$

(12) Find the angle between the surfaces $x^2 + y^2 + z^2 = 9$ $z = x^2 + y^2 - 2$ at $(2,-1,2)$

Soln: $\theta = \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right)$

(13) Find a and b such that the surfaces $ax^2 + byz = (a+2)x$ and $4x^2y + z^3 = 4$ cuts orthogonally at $(1,-1,2)$.

Soln:

Let $\varphi_1 = ax^2 + byz - (a+2)x$

$\varphi_2 = 4x^2y + z^3 - 4$

$$\nabla\varphi_1 = \vec{i}\frac{\partial}{\partial x}(ax^2 + byz - (a+2)x) + \vec{j}\frac{\partial}{\partial y}(ax^2 + byz - (a+2)x) + \vec{k}\frac{\partial}{\partial z}(ax^2 + byz - (a+2)x)$$

$$= \vec{i}(2ax - a - 2) + \vec{j}(bz) + \vec{k}(by)$$

$$\nabla\varphi_1\,(1,-1,2) = (a-2)\vec{i} + 2b\vec{j} - b\vec{k}$$

$$\nabla\varphi_2 = \vec{i}\frac{\partial}{\partial x}(4x^2y + z^3 - 4) + \vec{j}\frac{\partial}{\partial y}(4x^2y + z^3 - 4) + \vec{k}\frac{\partial}{\partial z}(4x^2y + z^3 - 4)$$

$$= \vec{i}(8xy) + \vec{j}(4x^2) + \vec{k}(3z^2)$$

$$\nabla\varphi_2\,(1,-1,2) = -8\vec{i} + 4\vec{j} + 12\vec{k}$$

Since the surfaces cut orthogonally,

$$\nabla\varphi_1 \cdot \nabla\varphi_2 = 0$$

$$[(a-2)\vec{i} + 2b\vec{j} - b\vec{k}] \cdot [-8\vec{i} + 4\vec{j} + 12\vec{k}] = 0$$

$$-8a + 16 - 8b + 12b = 0$$

$$-8a + 4b = -16$$

$$2a - b = 4 \longrightarrow (1)$$

Since the point $(1,-1,2)$ lies on φ,

$$a - 2b - (a+2) = 0 \Rightarrow \boxed{b = -1}$$

subs $b = -1$ in (1) $\Rightarrow \boxed{a = 3/2}$