Harmonic Function:

Any function which has continuous second order partial derivatives and which satisfies Laplace's Equation is called Harmonic function

i.e., $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$

(or) $\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2} = 0$

(or) $u_{xx} + u_{yy} = 0$

(or) $V_{xx} + V_{yy} = 0$.

2) Prove that $u = e^x \cos y$ is harmonic function.

Soln:

Given: $u = e^x \cos y$ — ①

Condition:

$$u_{xx} + u_{yy} = 0$$

Using ①

$$u_x = \cos y \cdot e^x$$

$$u_{xx} = e^x \cos y$$

and

$$u_y = e^x(-\sin y)$$

$$u_{yy} = -e^x(\cos y)$$

$$u_{xx} + u_{yy} = e^x \cos y + (-e^x \cos y)$$

$$= 0$$

$\therefore u = e^x \cos y$ is a harmonic function.

3) P.T. $u = \frac{1}{2}\log(x^2+y^2)$

Soln:

Given: $u = \frac{1}{2}\log(x^2+y^2)$ — ①

Condition:

$$u_{xx} + u_{yy} = 0.$$

Using ①,

$$\cancel{u_x = \frac{1}{2}\Big[}$$

$$u_x = \frac{1}{2} \frac{1}{x^2+y^2}(2x)$$

$$= \frac{x}{x^2+y^2}$$

$$u_{xx} = \frac{(x^2+y^2)(1) - x(2x)}{(x^2+y^2)^2}$$

$$= x^2+y^2-2x^2$$

$$= \frac{-x^2+y^2}{(x^2+y^2)^2} \quad \text{— ③}$$

Again

$$u_y = \frac{1}{2} \frac{1}{x^2+y^2}(2y)$$

$$= \frac{y}{x^2+y^2}$$

$$u_{yy} = \frac{(x^2+y^2)(1) - y(2y)}{(x^2+y^2)^2}$$

$$= \frac{x^2+y^2-2y^2}{(x^2+y^2)^2}$$

$$= \frac{+x^2-y^2}{(x^2+y^2)^2} \quad \text{— ④}$$

Hence from ③ & ④

$$u_{xx} + u_{yy} = 0.$$

$\therefore u$ is harmonic

4) P.T. $u = x^2 - y^2$

$v = \dfrac{-y}{x^2+y^2}$ is harmonic but $u+iv$ is not a regular.

Solution:

Given $u = x^2 - y^2$ —— (1)

$v = \dfrac{-y}{x^2+y^2}$ —— (2).

harmonic:

condition:

$u_{xx} + u_{yy} = 0$.

$v_{xx} + v_{yy} = 0$.

Using (1)

$u_x = \dfrac{\partial u}{\partial x} = 2x$

$u_{xx} = 2$

$u_y = \dfrac{\partial u}{\partial y} = -2y$

$u_{yy} = -2$

$u_{xx} + u_{yy} = 2 + (-2)$

$= 0$

Hence u is a harmonic function.

~~Using (2),~~

$$\cancel{v_x = \frac{(x^2+y^2)(}{(x^2+y^2)^2}}$$

Using (2),

$$v_x = -y\left(\frac{-1}{(x^2+y^2)^2}\right)(2x)$$

$$= \frac{2xy}{(x^2+y^2)^2}$$

$$v_{xx} = y\left[\frac{(x^2+y^2)^2(2) - 2x\,2(x^2+y^2)(2x)}{((x^2+y^2)^2)^2}\right]$$

$$= \frac{2y(x^2+y^2)^2 - 8x^2y(x^2+y^2)}{(x^2+y^2)^4}$$

$$= \frac{2y(x^2+y^2) - 8x^2y}{(x^2+y^2)^3}$$

$$= \frac{2yx^2 + 2y^3 - 8x^2y}{(x^2+y^2)^3}.$$

$$= \frac{-6x^2y + 2y^3}{(x^2+y^2)^3}$$

$$= \frac{+2y(x^2+y^2) - 8x^2y}{(x^2+y^2)^3}$$

$= -$

$$\boxed{v_{xx} = \frac{-6x^2y + 2y^3}{(x^2+y^2)^3}}$$ —— (5).

$$V_y = \frac{\partial u}{\partial y} = \frac{(x^2+y^2)(-1)-(-y)(2y)}{(x^2+y^2)^2}$$

$$= \frac{-(x^2+y^2)+2y^2}{(x^2+y^2)^2}$$

$$= \frac{-x^2-y^2+2y^2}{(x^2+y^2)^2}$$

$$V_y = \frac{-x^2+y^2}{(x^2+y^2)^2}$$

$$V_{yy} = \frac{(x^2+y^2)^2(2y)-(-x^2+y^2)2(x^2+y^2)(2y)}{(x^2+y^2)^4}$$

$$= \frac{(x^2+y^2)\left((x^2+y^2)2y - 4y(-x^2+y^2)\right)}{(x^2+y^2)^4}$$

$$= \frac{2x^2y+2y^3+4x^2y-4y^3}{(x^2+y^2)^3}$$

$$= \frac{6x^2y-2y^3}{(x^2+y^2)^3} \quad (6)$$

from (5) & (6).

$$V_{xx}+V_{yy}=0$$

Hence V is also a harmonic function.

Now consider,

$$f(z) = u+iv$$

$$= (x^2-y^2)+i\left(\frac{-y}{x^2+y^2}\right)$$

$$u_x = 2x$$

$$u_y = -2y$$

$$V_x = \frac{2xy}{(x^2+y^2)^2}$$

$$V_y = \frac{-(x^2+y^2)}{(x^2+y^2)^2}$$

since $u_x \neq V_y$

and $V_x \neq u_y$

the function $f(z) = u+iv$ is not regular

i.e., not analytic.