UNIT-3.

COMPLEX VARIABLES

Introduction:

If x and y are real numbers then $z = x + iy$ is called as complex number, where x is the real part of z and y is the imaginary part of z and the value of i is $\sqrt{-1}$

The complex number $x - iy$ is called as complex conjugate $(\bar{z} = x - iy)$ and it is denoted by $\bar{z}$

Note:

1) $|z| = \sqrt{x^2 + y^2}$
2) $|z^2| = z\bar{z}$
3) $z\bar{z} = x^2 + y^2 = r^2$
4) $|\bar{z}| = |z|$
5) Real part of $z = \frac{z + \bar{z}}{2}$
6) Im. part of $z = \frac{z + \bar{z}}{2i}$
7) $z = re^{i\theta}$ is called the polar form of z.

Functions of Complex Variable:

$w = f(z) = u(x, y) + iv(x, y)$ is a function of the complex variable $z = x + iy$ where $u(x, y)$ is the real part and $v(x, y)$ is the imaginary part of the complex function $f(z)$.

Analytic Function

Regular fun or Holomorphic fun.

A function defined at a point z_0 is said to be analytic at z_0, if it has a derivative at z_0 and at every point in some neighbourhood of z_0.

It is said to be analytic in a region R, if it is analytic at every point of R.

CAUCHY-RIEMANN EQUATIONS

Necessary conditions

for a complex function $f(z) = u(x,y) + iv(x,y)$ to be analytic in a region R, of the z plane, then,

(i) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$

i.e., $u_x = v_y$ and $v_x = -u_y$

provided $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x} \& \frac{\partial v}{\partial y}$ exists

Sufficient Condition

If the function $f(z) = u(x,y) + iv(x,y)$ is analytic in a region R of the z plane, if

(i) u_x, u_y, v_x and v_y exists and all are continuous

(ii) $u_x = v_y$ and $u_y = -v_x$

It is also called as C-R equations.

Polar form C-R Equs

Necessary conditions

If the function $w = f(w) = u(r,\theta) + iv(r,\theta)$ is analytic in a region R of the z-plane then,

(i) $\frac{\partial u}{\partial r}, \frac{\partial u}{\partial \theta}, \frac{\partial v}{\partial r}, \frac{\partial v}{\partial \theta}$ exists

(ii) $\frac{\partial u}{\partial r} = \frac{1}{r} \cdot \frac{\partial v}{\partial \theta}$ and

$$\frac{\partial v}{\partial r} = \frac{-1}{r} \cdot \frac{\partial u}{\partial \theta}.$$

Sufficient conditions.

(i) $\frac{\partial u}{\partial r}, \frac{\partial u}{\partial \theta}$ and $\frac{\partial v}{\partial r} \cdot \frac{\partial v}{\partial \theta}$ exists and are all continuous

(ii) $\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta}$

and $\frac{\partial v}{\partial r} = -\frac{1}{r} \frac{\partial u}{\partial \theta}$

Problems:

1) If $|z_1| = |z_2| = c$,

P.T $|z_1+z_2|^2 + |z_1-z_2|^2 = 4c^2$

Proof:

Since $|z|^2 = z\bar{z}$

$\Rightarrow |z_1+z_2|^2$

$= (z_1+z_2)\overline{(z_1+z_2)}$

$\Rightarrow |z_1-z_2|^2$

$= (z_1-z_2)\overline{(z_1-z_2)}$

LHS

$|z_1+z_2|^2 + |z_1-z_2|^2$

$= (z_1+z_2)\overline{(z_1+z_2)}$
$+ (z_1-z_2)\overline{(z_1-z_2)}$

$= (z_1+z_2)(\bar{z}_1+\bar{z}_2)$
$+ (z_1-z_2)(\bar{z}_1-\bar{z}_2)$

$= z_1\bar{z}_1 + z_1\bar{z}_2 + z_2\bar{z}_1 + z_2\bar{z}_2$
$+ z_1\bar{z}_1 - z_1\bar{z}_2 - z_2\bar{z}_1 + z_2\bar{z}_2$

$= 2z_1\bar{z}_1 + 2z_2\bar{z}_2$

$= 2|z_1|^2 + 2|z_2|^2$

$= 2(c)^2 + 2(c)^2$

$= 2c^2 + 2c^2 = 4c^2 = $ RHS

Hence proved.

2) Show that $f(z) = z^3$ is analytic in the entire z plane.

Solution:

Given $f(z) = z^3$

W.K.T $z = x+iy$

$f(z) = (x+iy)^3$

$= x^3 + 3x^2(iy) + 3x(iy)^2 + (iy)^3$

$= x^3 + 3ix^2y - 3xy^2 - iy^3$

$= x^3 - 3xy^2 + i(3x^2y - y^3)$

$= u + iv$

where $u = x^3 - 3xy^2$

$v = 3x^2y - y^3$

$u_x = \frac{\partial u}{\partial x} = 3x^2 - 3y^2$ — (1)

$u_y = \frac{\partial u}{\partial y} = 0 - 6xy$ — (2)

$v_x = \frac{\partial v}{\partial x} = 6xy - 0$ — (3)

$v_y = \frac{\partial v}{\partial y} = 3x^2 - 3y^2$ — (4)

from (1) & (4),

$u_x = v_y$

from (2) & (3)

$u_y = -v_x$.

ie, C-R eqns are satisfied

Hence $f(z) = u+iv$ is an analytic function.

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3) Prove that $w = z^2$ is analytic

Solution:

Given $f(z) = w = z^2$

W.K.T $z = x + iy$

$f(z) = (x+iy)^2$

$= x^2 + 2xiy + (iy)^2$

$= x^2 + 2ixy - 1y^2$

$= x^2 - y^2 + i(2xy)$

$= u + iv$

$u = x^2 - y^2$; $u_x = 2x$ | $u_y = -2y$

$v = 2xy$; $v_x = 2y$ | $v_y = 2x$

$u_x = v_y$ | $u_y = -v_x$

ie, C-R eqns are satisfied and Hence $f(z) = u+iv$ is an analytic function.

$f(z) = u + iv$

$u = r^n \cos n\theta$ | $v = r^n \sin n\theta$

$\frac{\partial u}{\partial r} = \cos n\theta \, n r^{n-1}$

$\frac{\partial u}{\partial \theta} = r^n(-\sin n\theta) \cdot n$

$\frac{\partial v}{\partial r} = \sin n\theta \, n r^{n-1}$

$\frac{\partial v}{\partial \theta} = r^n(\cos n\theta) \cdot n$

$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$

$\frac{\partial v}{\partial r} = \frac{-1}{r}\frac{\partial u}{\partial \theta}$

ie, C-R eqns in polar form are satisfied and hence $f(z) = u+iv$ is an analytic function.

4) Prove that $f(z) = z^n$ is analytic.

Soln:

Let $z = re^{i\theta}$

$f(z) = z^n$

$= (re^{i\theta})^n$

$= r^n (e^{i\theta})^n$

$= r^n e^{in\theta}$

$= r^n (\cos n\theta + i \sin n\theta)$

$f(z) = r^n \cos n\theta + i r^n \sin n\theta$

5) Determine whether the function $w = 2xy + i(x^2 - y^2)$ is analytic.

Soln: $w = u(x,y) + iv(x,y)$

$u = 2xy$ | $v = x^2 - y^2$

$u_x = 2y$ | $v_x = 2x$

$u_y = 2x$ | $v_y = -2y$

$u_x \neq u_y$ and $v_x = v_y$

C-R not satisfied

$\therefore$ w is not an analytic function.

6) ~~Det~~ Verify whether the function $f(z)=e^{-x}(\cos y - i\sin y)$ is analytic justify.

Solution:

Given:

$$f(z)=e^{-x}(\cos y - i\sin y)$$

$$u(x,y)+iv(x,y) = e^{-x}\cos y - e^{-x} i\sin y$$

$$u=e^{-x}\cos y \qquad v=-e^{-x}\sin y$$

$$\frac{\partial u}{\partial x} = -e^{-x}\cos y \qquad \frac{\partial v}{\partial x} = +e^{x}\sin y$$

$$\frac{\partial u}{\partial y} = -e^{-x}(-\sin y) \qquad \frac{\partial v}{\partial y} = (e^{x}\cos y)$$

$u_x = v_y$ and

$u_y = -v_x$

Since the C-R eqns are satisfied, $f(z)$ is an analytic function.