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NUMERICAL DIFFERENTIATION & INTEGRATION

NUMERICAL DIFFERENTIATION :

It is the process of computing the value of the derivative $\frac{dy}{dx}$ for some particular value of x, from the given data (x_i, y_i). If the values of x are equally spaced, we can use Newton's interpolation formula for equal intervals. If the values of x are unequally spaced, we can use Lagrange's interpolation formula (or) Newton's divided difference interpolation formula.

Differentiation using interpolation formulae:

Newton's forward difference formula to compute the derivatives :

Let us consider Newton's forward difference formula,

$$y = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y_0 + \cdots$$

where $u = \frac{x - x_0}{h}$

$$\text{i.e., } y = y_0 + u\Delta y_0 + \frac{(u^2-u)}{2}\Delta^2 y_0 + \frac{(u^3-3u^2+2u)}{6}\Delta^3 y_0 + \frac{(u^4-6u^3+11u^2-6u)}{24}\Delta^4 y_0 + \cdots \longrightarrow (1)$$

Now,

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{1}{h}\frac{dy}{du}$$

i.e., $$\frac{dy}{dx} = \frac{1}{h}\left\{ \Delta y_0 + \frac{(2u-1)}{2}\Delta^2 y_0 + \frac{(3u^2-6u+2)}{6}\Delta^3 y_0 + \frac{(4u^3-18u^2+22u-6)}{24}\Delta^4 y_0 + \cdots \right\} \rightarrow (2)$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left\{ \Delta^2 y_0 + (u-1)\Delta^3 y_0 + \frac{(6u^2-18u+11)}{12}\Delta^4 y_0 + \cdots \right\} \rightarrow (3)$$

$$\frac{d^3y}{dx^3} = \frac{1}{h^3}\left\{ \Delta^3 y_0 + \frac{12u-18}{12}\Delta^4 y_0 + \cdots \right\} \rightarrow (4)$$

In particular, at $x = x_0$, $u = 0$. Hence putting $u = 0$ in (2), (3) & (4) we get the values of first, second and third derivatives at $x = x_0$.

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \left(\frac{dy}{dx}\right)_{u=0} = \frac{1}{h}\left\{ \Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{\Delta^3 y_0}{3} - \frac{\Delta^4 y_0}{4} + \cdots \right\} \rightarrow (5)$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_0} = \left(\frac{d^2y}{dx^2}\right)_{u=0} = \frac{1}{h^2}\left\{ \Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \cdots \right\} \rightarrow (6)$$

$$\left(\frac{d^3y}{dx^3}\right)_{x=x_0} = \left(\frac{d^3y}{dx^3}\right)_{u=0} = \frac{1}{h^3}\left\{ \Delta^3 y_0 - \frac{3}{2}\Delta^4 y_0 + \cdots \right\} \rightarrow (7)$$

Newton's Backward Difference formula to compute the derivatives :

$$\frac{dy}{dx} = \frac{1}{h}\left\{ \nabla y_n + \frac{2v+1}{2}\nabla^2 y_n + \frac{(3v^2+6v+2)}{6}\nabla^3 y_n + \frac{4v^3+18v^2+22v+6}{24}\nabla^4 y_n + \cdots \right\}$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left\{ \nabla^2 y_n + (v+1)\nabla^3 y_n + \frac{6v^2+18v+11}{12}\nabla^4 y_n + \cdots \right\}$$

$$\frac{d^3y}{dx^3} = \frac{1}{h^3}\left\{ \nabla^3 y_n + \frac{12v+18}{12}\nabla^4 y_n + \cdots \right\}$$

In particular, at $x = x_n$, $v = 0$. Then

$$\left(\frac{dy}{dx}\right)_{x=x_n} = \frac{1}{h}\left\{ \nabla y_n + \frac{\nabla^2 y_n}{2} + \frac{\nabla^3 y_n}{3} + \frac{\nabla^4 y_n}{4} + \cdots \right\}$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_n} = \frac{1}{h^2}\left\{ \nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \cdots \right\}$$

$$\left(\frac{d^3y}{dx^3}\right)_{x=x_n} = \frac{1}{h^3}\left\{ \nabla^3 y_n + \frac{3}{2}\nabla^4 y_n + \cdots \right\}$$

Problems:

(1) The population of a certain town is given below. Find the rate of growth of the population in 1931, 1941, 1961 and 1971.

Year x	1931	1941	1951	1961	1971
Population in thousands y	40.62	60.80	79.95	103.56	132.65

Solution:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1931	40.62				
		20.18			
1941	60.80		-1.03		
		19.15		5.49	
1951	79.95		4.46		-4.47
		23.61		1.02	
1961	103.56		5.48		
		29.09			
1971	132.65				

(i) To get $f'(1931)$ and $f'(1941)$ we use forward formula.

$$u = \frac{x - x_0}{h} = \frac{1931 - 1931}{10} = 0$$

$$\left(\frac{dy}{dx}\right)_{u=0} = \frac{1}{h}\left\{\Delta y_0 - \frac{\Delta^2 y_0}{2} + \frac{\Delta^3 y_0}{3} - \frac{\Delta^4 y_0}{4} + \cdots\right\}$$

$$= \frac{1}{10}\left\{20.18 - \frac{(-1.03)}{2} + \frac{(5.49)}{3} - \frac{(-4.47)}{4}\right]$$

(3)

$$\left(\frac{dy}{dx}\right)_{u=0} = \frac{1}{10}\{20.18 + 0.515 + 1.83 + 1.1175\}$$

$$= \underline{2.3643}$$

(ii) To find $y'(1941)$:

$$u = \frac{x - x_0}{h} = \frac{1941 - 1931}{10} = 1$$

$$\frac{dy}{dx} = \frac{1}{h}\left\{\Delta y_0 + \frac{2u-1}{2}\Delta^2 y_0 + \frac{3u^2 - 6u + 2}{6}\Delta^3 y_0 + \frac{4u^3 - 18u^2 + 22u - 6}{24}\Delta^4 y_0 + \cdots\right\}$$

$$= \frac{1}{10}\left\{20.18 + \frac{1}{2}(-1.03) - \frac{1}{6}(5.49) + \frac{1}{12}(-4.47)\right\}$$

$$= \frac{1}{10}\{20.18 - 0.515 - 0.915 - 0.3725\}$$

$$= \underline{1.83775}$$

(iii) To find $y'(1961)$ and $y'(1971)$ we use Newton's backward formula

$$v = \frac{x - x_n}{h} = \frac{1961 - 1971}{10} = -1$$

$$\frac{dy}{dx} = \frac{1}{h}\left[\nabla y_n + \frac{2v+1}{2}\nabla^2 y_n + \frac{3v^2 + 6v + 2}{6}\nabla^3 y_n + \frac{4v^3 + 18v^2 + 22v + 6}{24}\nabla^4 y_n + \cdots\right]$$

$$= \frac{1}{10}\left[29.09 - \frac{1}{2}(5.48) - \frac{1}{6}(1.02) - \frac{1}{2}(-4.47)\right]$$

$$\frac{dy}{dx} = \frac{1}{10}\left\{29.09 - 2.74 - 0.17 + 0.3725\right\}$$

$$= \underline{2.6553}$$

(iii) To find $y'(1971)$.

$$v = \frac{x - x_n}{h} = \frac{1971 - 1971}{10} = 0$$

$$\left(\frac{dy}{dx}\right)_{v=0} = \frac{1}{h}\left\{\nabla y_n + \frac{\nabla^2 y_n}{2} + \frac{\nabla^3 y_n}{3} + \frac{\nabla^4 y_n}{4} + \cdots\right\}$$

$$= \frac{1}{10}\left\{29.09 + 2.74 + 0.34 - 1.1175\right\}$$

$$= \frac{1}{10}\left\{31.0525\right\}$$

$$= \underline{3.10525}$$

(4)

(2) A jet fighter's position on an aircraft carrier's runway was timed during landing.

t (sec) :	1.0	1.1	1.2	1.3	1.4	1.5	1.6
y (m) :	7.989	8.403	8.781	9.129	9.451	9.750	10.031

where y is the distance from the end of the carrier.
Estimate velocity $\left(\frac{dy}{dt}\right)$ and acceleration $\left(\frac{d^2y}{dt^2}\right)$ at
(i) $t = 1.1$ (ii) $t = 1.6$ using numerical differentiation.

Solution :

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
1.0	7.989						
		0.414					
1.1	8.403		−0.036				
		0.378		0.006			
1.2	8.781		−0.03		−0.002		
		0.348		0.004		0.001	
1.3	9.129		−0.026		−0.001		0.002
		0.322		0.003		0.003	
1.4	9.451		−0.023		0.002		
		0.299		0.005			
1.5	9.750		−0.018				
		0.281					
1.6	10.031						

(i) To find $t = 1.1$:

Here $h = 0.1$

$$u = \frac{x - x_0}{h} = \frac{1.1 - 1.0}{0.1} = 1$$

$$\left(\frac{dy}{dx}\right)_{t=1.1} = \left(\frac{dy}{dx}\right)_{u=1} = \frac{1}{h}\left[\Delta y_0 + \left(\frac{2u-1}{2}\right)\Delta^2 y_0 + \left(\frac{3u^2-6u+2}{6}\right)\Delta^3 y_0 + \left(\frac{4u^3-18u^2+22u-6}{24}\right)\Delta^4 y_0 + \cdots\right]$$

$$\frac{dy}{dx} = \frac{1}{0.1}\left[0.414 + \left(\frac{2-1}{2}\right)(-0.036) + \left(\frac{3-6-2}{6}\right)(0.006) + \left(\frac{4-18+22-6}{24}\right)(-0.002) + \cdots\right]$$

$$= \underline{3.9483}$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\Delta^2 y_0 + (u-1)\Delta^3 y_0 + \left(\frac{6u^2-18u+11}{12}\right)\Delta^4 y_0 + \cdots\right]$$

$$= \frac{1}{(0.1)^2}\left[-0.036 + (1-1)(0.006) + \left(\frac{6-18+11}{12}\right)(-0.002)\cdots\right]$$

$$= \underline{-3.5833}$$

(ii) To find $t = 1.6$:

$$v = \frac{t-t_n}{h} = \frac{1.6-1.6}{0.1} = 0$$

$$\frac{dy}{dx} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{\nabla^4 y_n}{4} + \cdots\right]$$

$$= \frac{1}{0.1}\left[0.281 + \frac{1}{2}(-0.018) + \frac{1}{3}(0.005) + \frac{1}{4}(0.002) + \frac{1}{5}(0.003) + \frac{1}{6}(0.002)\right]$$

$$= \underline{2.751}$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \cdots\right]$$

$$= \frac{1}{(0.1)^2}\left[-0.018 + 0.005 + \frac{11}{12}(0.002) + \cdots\right]$$

$$= \underline{-1.1167}$$

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(3) Using the following data, find $f'(5)$, $f''(5)$ and the maximum value of $f(x)$

x:	0	2	3	4	7	9
$f(x)$:	4	26	58	112	466	922

Solution: Since the values of x are not equally spaced, we use Newton's divided difference formula

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
0	4				
		$\frac{26-4}{2} = 11$			
2	26		$\frac{32-11}{3-0} = 7$		
		$\frac{58-26}{3-2} = 32$		$\frac{11-7}{4-0} = 1$	
3	58		$\frac{54-32}{4-2} = 11$		0
		$\frac{112-58}{4-3} = 54$		$\frac{16-11}{7-2} = 1$	
4	112		$\frac{118-54}{7-3} = 16$		0
		$\frac{466-112}{7-4} = 118$		$\frac{22-16}{9-3} = 1$	
7	466		$\frac{228-118}{9-4} = 22$		
		$\frac{922-466}{9-7} = 228$			
9	922				

By Newton's divided difference formula,

$$f(x) = f(x_0) + (x-x_0)\Delta f(x_0) + (x-x_0)(x-x_1)\Delta^2 f(x_0) + (x-x_0)(x-x_1)(x-x_2)\Delta^3 f(x_0) + \cdots$$

$$= 4 + (x-0)\,11 + (x-0)(x-2)(7) + (x-0)(x-2)(x-3)(1)$$

$$f(x) = x^3 + 2x^2 + 3x + 4$$

$$f'(x) = 3x^2 + 4x + 3$$

$$f''(x) = 6x + 4$$

$$f'(5) = 3(5^2) + 4(5) + 3 = 98$$

$$f''(5) = 6(5) + 4 = 34$$

$f(x)$ is maximum if $f'(x) = 0$

i.e., $3x^2 + 4x + 3 = 0$

But the roots of this equation are imaginary. Hence there is no extremum value.

(4) From the following table, find the value of x for which y is minimum and find this value of y.

x :	-2	-1	0	1	2	3	4
y :	2	-0.25	0	-0.25	2	15.75	56

Solution :

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
-2	2					
		-2.25				
-1	-0.25		2.50			
		0.25		-3		
0	0		-0.50		6	
		-0.25		3		0
1	-0.25		2.50		6	
		2.25		9		0
2	2		11.50		6	
		13.75		15		
3	15.75		26.50			
		40.25				
4	56					

Here $h = 1$.

For minimum value of y, $\frac{dy}{dx} = 0$

i.e., $\frac{1}{h}\left[\Delta y_0 + \left(\frac{2u-1}{2}\right)\Delta^2 y_0 + \cdots\right] = 0$

i.e., $\frac{1}{1}\left[-2.25 + \left(\frac{2u-1}{2}\right)(2.5)\right] = 0$

$$\boxed{u = 1.4}$$

$\therefore \quad \frac{x - x_0}{h} = 1.4 \Rightarrow x = 1.4h + x_0 = \underline{-0.6}$

$y(x = -0.6) = y_0 + \frac{u\Delta y_0}{1!} + \frac{u(u-1)}{2!}\Delta^2 y_0 + \cdots$

$= 2 + (1.4)(-2.25) + \frac{(1.4)(0.4)(2.5)}{2} + \frac{(1.4)(0.4)(-0.6)(-3)}{6} + \frac{(1.4)(0.4)(-0.6)(-1.6)6}{24}$

$= \underline{-0.1476}$