

(An Autonomous Institution) DEPARTMENT OF MATHEMATICS



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NUMERICAL DIFFERENTIATION & INTEGRATION

NUMERICAL DIFFERENTIATION:

It is the process of computing the value of the desivative dy for some particular value of x, from the given data (x;, yi). If the values of x are equally spaced, we can use Newton's interpolation formula for equal intervals. If the values of x are unequally spaced, we can use Lagrange's interpolation formula (or) Newton's divided difference interpolation formula.

Differentiation using interpolation formulae:

Newton's forward difference formula to compute the derivatives:

Let us consider Newton's forward difference

formula,

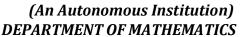
$$u(u-1)(u-2)(u-3) \Delta^{4} y_{0} + \cdots$$

where $u = \frac{\chi - \chi_0}{h}$

Now,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$







$$\frac{dy}{dx} = \frac{1}{h} \frac{dy}{du}$$
i.e.
$$\frac{dy}{dx} = \frac{1}{h} \left\{ \Delta y_0 + \frac{(2u-1)}{2} \Delta^2 y_0 + \frac{(3u^2 - 6u + a)}{6} \Delta^2 y_0 + \frac{3u^2 - 6u + a}{6} \Delta^2 y_0 + \frac$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2} \left\{ \Delta^2 y_0 + (u-1)\Delta^3 y_0 + (6u^2 - 18u + 11) \Delta^4 y_0 + \frac{1}{3} - \frac{1}{3} \right\}$$

$$\frac{d^3y}{dx^3} = \frac{1}{h^3} \left\{ \Delta^3 y_0 + \frac{12u - 18}{12} \Delta^4 y_0 + \cdots \right\} \longrightarrow \textcircled{F}$$

In particular, at $x = x_0$, u = 0. Hence putting u = 0 in (2), (3) & (4) we get the values of first, Second and third derivatives at $z = x_0$.

$$\left(\frac{dy}{dx}\right)_{\chi=\chi_{0}} = \left(\frac{dy}{dx}\right)_{u=0} = \frac{1}{h} \left\{ \Delta y_{0} - \frac{1}{2} \Delta^{2} y_{0} + \frac{\Delta^{3} y_{0}}{3} - \frac{\Delta^{4} y_{0} + \cdots y_{0}}{4} - \frac{\Delta^{4} y_{0} + \cdots y_{0}}{4} \right\}$$

$$\left(\frac{d^{2}y}{dx^{2}}\right)_{\chi=\chi_{0}} = \left(\frac{d^{2}y}{dx^{2}}\right)_{u=0} = \frac{1}{h^{2}} \left\{ \Delta^{2} y_{0} - \Delta^{3} y_{0} + \frac{11}{12} \Delta^{4} y_{0} - \cdots y_{0} - \frac{1}{12} \Delta^{4} y_{0} - \cdots y_{0} \right\}$$

$$\frac{d^{2}y}{dx^{2}} = \left(\frac{d^{2}y}{dx^{2}}\right)_{u=0} = \frac{1}{h^{2}} \left\{ \Delta^{2}y_{0} - \Delta^{3} y_{0} + \frac{11}{12} \Delta^{4} y_{0} - \cdots y_{0} - \frac{1}{12} \Delta^{4} y_{0} - \cdots y_{0} \right\}$$

$$\left(\frac{d^3y}{dx^3}\right)_{x=x_0} = \left(\frac{d^3y}{dx^3}\right)_{u=0} = \frac{1}{h^3} \left\{ \Delta^3y_0 - \frac{3}{2} \Delta^4y_0 + \cdots \right\}$$

$$\longrightarrow \textcircled{?}$$



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Newton's Backward Difference formula to compute the

$$\frac{dy}{dx} = \frac{1}{h} \left\{ \nabla y_n + \frac{2v+1}{2} \nabla^2 y_n + \frac{(3v^2 + 6v + 2)}{6} \nabla^3 y_n + \frac{4v^3 + 18v^2 + 22v + 6}{24} \nabla^4 y_n + \cdots \right\}$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2} \left\{ \nabla^2 y_n + (v+1) \nabla^3 y_n + \frac{6v^2 + 18v + 11}{12} \nabla^4 y_n + \cdots \right\}$$

$$\frac{d^3y}{da^3} = \frac{1}{h^3} \left\{ \nabla^3 y_n + \frac{12v + 18}{12} \nabla^4 y_n + \cdots \right\}$$

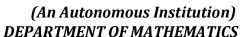
In particular, at $x = x_n$, v = 0. Then

$$\left(\frac{dy}{dx}\right)_{\chi=\chi_n} = \frac{1}{h} \left\{ \nabla y_n + \frac{\nabla^2 y_n}{a} + \frac{\nabla^3 y_n}{3} + \frac{\nabla^4 y_n}{4} + \cdots \right\}$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_n} = \frac{1}{h^2} \left\{ \nabla^2 y_n + \nabla^3 y_n + \frac{11}{12} \nabla^4 y_n + \cdots \right\}$$

$$\left(\frac{d^3y}{dx^3}\right)_{\chi=\chi_n} = \frac{1}{h^3} \left\{ \nabla^3 y_n + \frac{3}{2} \nabla^4 y_n + \cdots \right\}$$







Problems:

1) The population of a certain town is given below. Find the rate of growth of the population in 1931.

Year: 1931 1941 1951 1961 1971

Population : 40.62 60.80 79.95 103.56 132.65 in thousands y

Solution:

$$\chi$$
 y Δy $\Delta^{2}y$ $\Delta^{3}y$ $\Delta^{4}y$

1931 40.62

1941 60.86

19.15

19.15

19.15

23.61

19.61

103.56

29.09

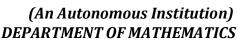
1971 132.65

(i) To get f'(1931) and f'(1941) we use forward formula.

$$U = \frac{\chi - \chi_0}{h} = \frac{1931 - 1931}{10} = 0$$

$$\left(\frac{dy}{dx}\right)_{u=0} = \frac{1}{h} \left\{ \Delta y_0 - \frac{\Delta^2 y_0}{2} + \frac{\Delta^3 y_0}{3} - \frac{\Delta^4 y_0}{4} + \cdots \right\}$$
$$= \frac{1}{10} \left\{ 20.18 - \left(\frac{-1.03}{2}\right) + \frac{(5.49)}{3} - \frac{(-4.47)}{4} \right\}$$







$$\left(\frac{dy}{dx}\right)_{u=0} = \frac{1}{10} \left\{ 20.18 + 0.515 + 1.83 + 1.1175 \right\}$$
$$= 2.3643$$

(ii) To find y'(1941):
$$U = \frac{x - x_0}{h} = \frac{1941 - 1931}{10} = 1$$

$$\frac{dy}{dx} = \frac{1}{h} \left\{ \Delta y_{o} + \frac{2u-1}{a} \Delta^{2} y_{o} + \frac{3u^{2}-6u+2}{6} \Delta^{3} y_{o} + \frac{4u^{3}-18u^{2}+22u-6}{24} \Delta^{4} y_{o} + \cdots \right\}$$

$$= \frac{1}{10} \left\{ 20.18 + \frac{1}{2} (-1.03) - \frac{1}{6} (5.49) + \frac{1}{12} (-4.47) \right\}$$

$$= \frac{1}{10} \left\{ 20.18 - 0.515 - 0.915 - 0.3735 \right\}$$

$$= 1.83775$$

(iii) To find y'(1961) and y'(1971) we use Newton's backward formula

$$V = \frac{x - x_n}{h} = \frac{1961 - 1971}{10} = -1$$

$$\frac{dy}{dx} = \frac{1}{h} \left[\nabla y_n + \frac{2v+1}{2} \nabla^2 y_n + \frac{3v^2 + 6v + 2}{6} \nabla^3 y_n + \frac{3v^2 + 6v + 2}{6} \nabla^4 y_n + \cdots \right]$$

$$= \frac{1}{10} \left[a9.09 - \frac{1}{2} (5.48) - \frac{1}{6} (1.02) - \frac{1}{2} (-4.47) \right]$$



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$$\frac{dq}{dx} = \frac{1}{10} \left\{ 39.09 - 2.74 - 0.17 + 0.3735 \right\}$$

$$= 2.6553$$

$$V = \frac{x - x_0}{h} = \frac{1971 - 1971}{10} = 0$$

$$\left(\frac{dy}{dx}\right)_{k=0} = \frac{1}{h} \left\{ \nabla y_0 + \frac{\nabla^2 y_0}{2} + \frac{\nabla^3 y_0}{3} + \frac{\nabla^3 y_0}{4} + \cdots \right\}$$

$$= \frac{1}{10} \left\{ 39.09 + 2.74 + 0.34 - 1.1175 \right\}$$

$$= \frac{1}{10} \left\{ 31.0535 \right\}$$

$$= 3.10535$$



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(2) A jet fighter's position on an aircraft carrier's runway was timed during landing.

1.2 1.3 1.4 1.5 1.6 t(sec): 1.0 1.1 y (m): 7.989 8.403 8.781 9.129 9.451 9.750 10.031

Where y is the distance from the end of the carrier. Estimate velocity $\left(\frac{dy}{dt}\right)$ and acceleration $\left(\frac{d^2y}{dt^2}\right)$ at

(i) t = 1.1 (ii) t = 1.6 using numerical differentiation.

Solution:

Solution:

$$\alpha$$
 y Δy $\Delta^2 y$ $\Delta^3 y$ $\Delta^4 y$ $\Delta^5 y$ $\Delta^6 y$

7.989 1.0

6.414

(i) To find t=1.1:

$$u = \frac{x - x_0}{h} = \frac{1 \cdot 1 - 1 \cdot 0}{0 \cdot 1} = 1$$

$$\left(\frac{dy}{dx}\right)_{t=1\cdot 1} = \left(\frac{dy}{dx}\right)_{u=1} = \frac{1}{h} \left[\Delta y_0 + \left(\frac{2u-1}{2}\right)\Delta^2 y_0 + \left(\frac{3u^2-6u+2}{6}\right)\Delta^3 y_0 + \left(\frac{4u^3-18u^2+22u-6}{24}\right)\Delta^4 y_0 + \cdots\right]$$







$$\frac{dy}{dx} = \frac{1}{0.1} \left[0.414 + \left(\frac{2-1}{2} \right) (-0.036) + \left(\frac{3-6-2}{6} \right) (0.006) \right] + \left(\frac{4-18+23-6}{24} \right) (-0.002) + \cdots \right]$$

$$= \frac{3.9483}{dx^2} = \frac{1}{h^2} \left[\Delta^2 y_0 + (u-1) \Delta^3 y_0 + \left(\frac{6u^2-18u+11}{12} \right) \Delta^4 y_0 + \cdots \right]$$

$$= \frac{1}{(0.1)^2} \left[-0.036 + (1-1) (0.006) + \left(\frac{6-18+11}{12} \right) (-0.002) \right]$$

$$= -3.5833$$
(ii) To kind $t = 1.6$:
$$V = \frac{t-t_n}{h} = \frac{1.6-1.6}{0.1} = 0$$

$$\frac{dy}{dx} = \frac{1}{h} \left[\nabla y_n + \frac{1}{2} \nabla^2 y_n + \frac{1}{3} \nabla^3 y_n + \frac{\nabla^4 y_n}{4} + \cdots \right]$$

$$= \frac{1}{0.1} \left[0.281 + \frac{1}{2} (-0.018) + \frac{1}{3} (0.005) + \frac{1}{4} (0.002) + \frac{1}{15} (0.003) + \frac{1}{16} (0.002) \right]$$

$$= \frac{2.751}{dx^2} \left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12} \nabla^4 y_n + \cdots \right]$$

$$= \frac{1}{(0.02)^2} \left[-0.018 + 0.005 + \frac{11}{12} (0.002) + \cdots \right]$$

= -1.1167



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(3) Using the following data, find
$$f'(5)$$
, $f''(5)$ and the maximum value of $f(x)$
 $x: 0 = 2 = 3 = 4 = 7$
 $f(x): 4 = 36 = 58 = 113 = 466 = 933$

Solution: Since the values of x are not equally spaced, we use Newton's divided difference formula

 $x = f(x) = A f(x) = A f(x) = A^3 f(x) = A^4 f(x)$
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From the following table, find the value of
$$x$$
 for which y is minimum and find this value of y .

$$x: -2 -1 0 1 2 3 4$$

 $y: 2 -0.25 0 -0.25 2 15.75 56$

Solution:

Here h = 1.

For minimum value of y,
$$\frac{dy}{dx} = 0$$

i.e.,
$$\frac{1}{h} \left[\Delta y_0 + \left(\frac{2u-1}{2} \right) \Delta^2 y_0 + \cdots \right] = 0$$

i.e.,
$$\frac{1}{1} \left[-2.25 + \left(\frac{2u-1}{2} \right) (2.5) \right] = 0$$

$$\frac{x - x_0}{h} = 1.4 \implies x = 1.4h + x_0 = -0.6$$

$$y(x = -0.6) = y_0 + u \Delta y_0 + u(u-1) \Delta^2 y_0 + \cdots$$

$$= 2 + (1.4)(-2.25) + (1.4)(0.4)(2.5) + (1.4)(0.4)$$

$$= -0.1476 + (1.4)(0.4)(-0.6)(-16)6$$

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