FOURTH ORDER RUNGE-KUTTA METHOD FOR SOLVING FIRST AND SECOND ORDER EQUATIONS :

Second order R-K method:

$$k_1 = h f(x,y)$$

$$k_2 = h f\left(x+\frac{h}{2}, y+\frac{k_1}{2}\right)$$

$$\Delta y = k_2 \quad \text{where } h = \Delta x$$

Third order R-K method:

$$k_1 = h f(x,y)$$

$$k_2 = h f\left(x+\frac{h}{2}, y+\frac{k_1}{2}\right)$$

$$k_3 = h f\left(x+h, y+2k_2-k_1\right)$$

and $$\Delta y = \frac{1}{6}(k_1 + 4k_2 + k_3)$$

Fourth order R.K method:

$$k_1 = h f(x,y)$$

$$k_2 = h f\left(x+\frac{h}{2}, y+\frac{k_1}{2}\right)$$

$$k_3 = h f\left(x+\frac{h}{2}, y+\frac{k_2}{2}\right)$$

$$k_4 = h f(x+h, y+k_3)$$

and $$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$y(x+h) = y(x) + \Delta y$$

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Problems :

(1) Using R.K method of 4^{th} order, Solve $\frac{dy}{dx} = \frac{y^2 - x^2}{y^2 + x^2}$ with $y(0) = 1$ at $x = 0.2$ & $x = 0.4$ with $h = 0.2$

Solution :

Given : $x_0 = 0$, $y_0 = 1$, $x_1 = 0.2$, $h = 0.2$

$$f(x,y) = \frac{y^2 - x^2}{y^2 + x^2}$$

$$K_1 = h f(x_0, y_0) = 0.2 \left[\frac{y_0^2 - x_0^2}{y_0^2 + x_0^2}\right] = 0.2 \left[\frac{1-0}{1+0}\right] = \underline{0.2}$$

$$K_2 = hf\left[x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right] = (0.2) f \left[0 + \frac{0.2}{2}, 1 + \frac{0.2}{2}\right]$$

$$= (0.2) f(0.1, 1.1) = (0.2) \left[\frac{(1.1)^2 - (0.1)^2}{(1.1)^2 + (0.1)^2}\right]$$

$$= (0.2) \left[\frac{1.2}{1.22}\right] = (0.2)(0.9836)$$

$$K_2 = \underline{0.19672}$$

$$K_3 = hf\left[x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right]$$

$$= (0.2) f \left[0 + \frac{0.2}{2}, 1 + \frac{0.19672}{2}\right]$$

$$= (0.2) f(0.1, 1.09836)$$

$$= (0.2) \left[\frac{(1.0984)^2 - (0.1)^2}{(1.0984)^2 + (0.1)^2}\right] = \underline{0.1967}$$

$$K_4 = hf(x_0 + h, y_0 + K_3)$$

$$= (0.2) f(0.2, 1.1967)$$

$$= (0.2) \left[\frac{(1.1967)^2 - (0.2)^2}{(1.1967)^2 + (0.2)^2}\right]$$

$$K_4 = \underline{0.1891}$$

$$\Delta y = \frac{1}{6}\left[K_1 + 2K_2 + 2K_3 + K_4\right]$$

$$= \frac{1}{6}\left[0.2 + 2(0.19672) + 2(1.1967) + 0.1891\right]$$

$$\boxed{\Delta y = 0.19598}$$

$$\therefore y(0.2) = y_1 = y_0 + \Delta y = 1 + 0.19598 = \underline{1.19598}$$

To find $y(0.4)$

$x_1 = 0.2,\ y_1 = 1.19598$

$K_1 = 0.1891 \mid K_3 = 0.1793$

$K_2 = 0.1794 \mid K_4 = 0.1687$

$\Delta y = 0.1792$

$y_2 = y_1 + \Delta y = 1.3751$

(2) Find $y(0.7)$ and $y(0.8)$ given that $y' = y - x^2$, $y(0.6) = 1.7379$ by using Runge-Kutta method of fourth order. Take $h = 0.1$

Solution:

Given : $f(x,y) = y - x^2$

$x_0 = 0.6\ ,\ y_0 = 1.7379$

$h = 0.1\ ,\ x_1 = 0.7,\ x_2 = 0.8$

$$K_1 = hf(x_0, y_0) = (0.1)\left[y_0 - x_0^2\right]$$

$$= (0.1)\left[1.7379 - (0.6)^2\right]$$

$$K_1 = \underline{0.13779}$$

$$K_2 = hf\left[x_0 + \frac{h}{2},\ y_0 + \frac{K_1}{2}\right] = (0.1)\, f\left[0.6 + \frac{0.1}{2},\ 1.7379 + \frac{0.1378}{2}\right]$$

$$= 0.1\, f\left[0.65,\ 1.8068\right]$$

$$= (0.1)\left[1.8068 - (0.65)^2\right] = \underline{0.1384}$$

$$K_3 = hf\left[x_0 + \frac{h}{2},\ y_0 + \frac{K_2}{2}\right] = (0.1)\, f\left[0.65,\ 1.7379 + \frac{0.13843}{2}\right]$$

$$= (0.1)\, f\left[0.65,\ 1.807115\right]$$

$$= (0.1)\left[1.807115 - (0.65)^2\right]$$

$$= \underline{0.13846}$$

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$$K_4 = h f [x_0 + h, y_0 + K_3]$$
$$= (0.1) f [0.6 + 0.1, 1.7379 + 0.13846]$$
$$= (0.1) f [0.7, 1.87636]$$
$$= (0.1) [1.87636 - (0.7)^2] = \underline{0.13864}$$

$$\Delta y = \frac{1}{6} [K_1 + 2K_2 + 2K_3 + K_4]$$
$$= \frac{1}{6} [0.13779 + 2(0.13843) + 2(0.13846) + 0.13864]$$
$$= \underline{0.13837}$$

$$y_1 = y(0.7) = y_0 + \Delta y = 1.7379 + 0.13837 = \underline{1.8763}$$
$$\boxed{y(0.7) = 1.8763}$$

Here $x_1 = 0.7$, $y_1 = 1.8763$

$$K_1 = h f(x_1, y_1) = 0.1 f[0.7, 1.8763]$$
$$= (0.1)[1.8763 - (0.7)^2] = \underline{0.1386}$$

$$K_2 = h f\left[x_1 + \frac{h}{2}, y_1 + \frac{K_1}{2}\right] = (0.1) f\left[0.7 + \frac{0.1}{2}, 1.8763 + \frac{0.1386}{2}\right]$$
$$= 0.1 f [0.75, 1.9453]$$
$$= (0.1) [1.9453 - (0.75)^2]$$
$$= \underline{0.13828}$$

$$K_3 = h f\left[x_1 + \frac{h}{2}, y_1 + \frac{K_2}{2}\right]$$
$$= (0.1) f\left[0.75, 1.876 + \frac{0.13828}{2}\right]$$
$$= (0.1) f [0.75, 1.94514]$$
$$= (0.1) [1.94514 - (0.75)^2]$$
$$= \underline{0.1383}$$

$$k_4 = h\, f\,[x_1 + h,\ y_1 + k_3]$$

$$= (0.1)\, f\,[0.7 + 0.1,\ 1.876 + 0.1383]$$

$$= (0.1)\, f\,[0.8,\ 2.0143]$$

$$= (0.1)\,[(2.0143) - (0.8)^2]$$

$$= \underline{0.1374}$$

$$\Delta y = \frac{1}{6}\,[k_1 + 2k_2 + 2k_3 + k_4]$$

$$= \frac{1}{6}\,[0.1386 + 2(0.1383) + 2(0.1383) + 0.1374]$$

$$= \underline{0.1382}$$

$$y_2 = y(0.8) = y_1 + \Delta y = 1.876 + 0.1382 = 2.0142$$

$$\boxed{y(0.8) = 2.0142}$$

$$\Delta y = \frac{1}{6}\left[K_1 + 2K_2 + 2K_3 + K_4\right]$$

$$= \frac{1}{6}\left[0.2 + 2(0.19672) + 2(1.1967) + 0.1891\right]$$

$$\boxed{\Delta y = 0.19598}$$

$$\therefore y(0.2) = y_1 = y_0 + \Delta y = 1 + 0.19598 = \underline{1.19598}$$

To find $y(0.4)$

$x_1 = 0.2$, $y_1 = 1.19598$

$K_1 = 0.1891$ | $K_3 = 0.1793$

$K_2 = 0.1794$ | $K_4 = 0.1687$

$\Delta y = 0.1792$

$y_2 = y_1 + \Delta y = 1.3751$

(2) Find $y(0.7)$ and $y(0.8)$ given that $y' = y - x^2$, $y(0.6) = 1.7379$ by using Runge-Kutta method of fourth order. Take $h = 0.1$

Solution:

Given : $f(x,y) = y - x^2$

$x_0 = 0.6$, $y_0 = 1.7379$

$h = 0.1$, $x_1 = 0.7$, $x_2 = 0.8$

$$K_1 = h f(x_0, y_0) = (0.1)\left[y_0 - x_0^2\right]$$

$$= (0.1)\left[1.7379 - (0.6)^2\right]$$

$$K_1 = \underline{0.13779}$$

$$K_2 = hf\left[x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right] = (0.1) f\left[0.6 + \frac{0.1}{2}, 1.7379 + \frac{0.1378}{2}\right]$$

$$= 0.1 f\left[0.65, 1.8068\right]$$

$$= (0.1)\left[1.8068 - (0.65)^2\right] = \underline{0.1384}$$

$$K_3 = hf\left[x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right] = (0.1) f\left[0.65, 1.7379 + \frac{0.13843}{2}\right]$$

$$= (0.1) f\left[0.65, 1.807115\right]$$

$$= (0.1)\left[1.807115 - (0.65)^2\right]$$

$$= \underline{0.13846}$$

RUNGE-KUTTA METHOD FOR SECOND ORDER DIFFERENTIAL EQUATIONS:

Find the Solution of $y'' = f(x, y, y')$ given $y(x_0) = y_0$, $y'(x_0) = y_0'$.

Now set $y' = z$ and $y'' = z'$

Hence, differential equation reduces to

$$\frac{dy}{dx} = y' = z.$$

and $\frac{dz}{dx} = z' = y'' = f(x, y, y') = f(x, y, z)$

$\therefore$ $\frac{dy}{dx} = z$ and $\frac{dz}{dx} = f(x, y, z)$ are Simultaneous equations where $f_1(x, y, z) = z$ and $f_2(x, y, z) = f(x, y, z)$ given. Also $y(0)$ and $z(0)$ are given.

(1) Solving the system of differential equations $\frac{dy}{dx} = xz+1$, $\frac{dz}{dx} = -xy$ for $x = 0.3$ using fourth order R-K method, the initial values are $x=0$, $y=0$, $z=1$

Solution:

Given: $x_0 = 0$, $y_0 = 0$, $z_0 = 1$, $h = 0.3$

$f_1(x,y,z) = xz+1$	$f_2(x,y,z) = -xy$
$K_1 = h f_1(x_0, y_0, z_0)$ $= (0.3)[x_0 z_0 + 1]$ $= (0.3)(0+1) = 0.3$ $K_1 = 0.3$	$l_1 = h f_2(x_0, y_0, z_0)$ $= (0.3)[-x_0 y_0]$ $= (0.3)(0)$ $l_1 = 0$
$K_2 = h f_1\left[x_0+\frac{h}{2}, y_0+\frac{K_1}{2}, z_0+\frac{l_1}{2}\right]$ $= (0.3) f_1\left[0+\frac{0.3}{2}, 0+\frac{0.3}{2}, 1+\frac{0}{2}\right]$ $= (0.3) f_1[0.15, 0.15, 1]$ $= (0.3)[(0.15)(1)+1]$ $= 0.345$	$l_2 = h f_2\left[x_0+\frac{h}{2}, y_0+\frac{K_1}{2}, z_0+\frac{l_1}{2}\right]$ $= (0.3) f_2\left[0+\frac{0.3}{2}, 0+\frac{0.3}{2}, 1+\frac{0}{2}\right]$ $= (0.3) f_2[0.15, 0.15, 1]$ $= (0.3)[-(0.15)(0.15)]$ $= -0.007$
$K_3 = h f_1\left[x_0+\frac{h}{2}, y_0+\frac{K_2}{2}, z_0+\frac{l_2}{2}\right]$ $= (0.3) f_1\left[0.15, 0+\frac{0.345}{2}, 1-\frac{0.007}{2}\right]$ $= (0.3) f_1[0.15, 0.1725, 0.9965]$ $= (0.3)[(0.15)(0.9965)+1]$ $= 0.3448$	$l_3 = h f_2\left[x_0+\frac{h}{2}, y_0+\frac{K_2}{2}, z_0+\frac{l_2}{2}\right]$ $= (0.3) f_2\left[0.15, 0+\frac{0.345}{2}, 1-\frac{0.007}{2}\right]$ $= (0.3) f_2[0.15, 0.1725, 0.9965]$ $= (0.3)[-(0.15)(0.1725)]$ $= -0.0078$

$$K_4 = h f_1 [x_0+h,\ y_0+k_3,\ z_0+l_3]$$
$$= (0.3) f_1 [0.03, 0.3448, 1-0.0078]$$
$$= (0.3) [(0.3)(0.9922)+1]$$
$$= 0.3893$$

$$l_4 = h f_2 [x_0+h,\ y_0+k_3,\ z_0+l_3]$$
$$= (0.3) f_2 [0.3, 0.3448, 0.9922]$$
$$= (0.3) [-(0.3)(0.3448)]$$
$$= -0.031032$$

$$\Delta y = \frac{1}{6} [K_1 + 2K_2 + 2K_3 + K_4]$$
$$= \frac{1}{6} [0.3 + 2(0.345) + 2(0.3448) + 0.3893)]$$
$$= \frac{1}{6} (2.0689$$
$$= 0.34482$$

$$\Delta z = \frac{1}{6} [l_1 + 2l_2 + 2l_3 + l_4]$$
$$= \frac{1}{6} [0 + 2(-0.007) + 2(-0.0078) + (-0.031032)]$$
$$= \frac{-1}{6} (0.060632)$$
$$= -0.01011$$

$$y_1 = y_0 + \Delta y$$
$$= 0 + 0.34482$$
$$y(0.3) = 0.34482$$

$$z_1 = z_0 + \Delta z$$
$$= 1 - 0.01011$$
$$z(0.3) = 0.9899$$

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(1) Consider the second order initial value problem $y'' - 2y' + 2y = e^{2t}\sin t$ with $y(0) = -0.4$ and $y'(0) = -0.6$ using fourth order R.K method, find $y(0.2)$.

Solution:

Let $t = x$.

$$y'' - 2y' + 2y = e^{2x}\sin x$$

$$y(0) = -0.4,\ y'(0) = -0.6,\ h = 0.2$$

Setting $y' = z$ the equation becomes

$$z' = 2z - 2y + e^{2x}\sin x$$

$$f_1(x,y,z) = \frac{dy}{dx} = z,\quad f_2(x,y,z) = \frac{dz}{dx} = 2z - 2y + e^{2x}\sin x$$

Given: $y_0 = -0.4,\ z_0 = y_0' = -0.6,\ x_0 = 0$.

$$K_1 = h f_1(x_0, y_0, z_0) = (0.2)(z_0) = (0.2)(-0.6) = -0.12$$

$$l_1 = h f_2(x_0, y_0, z_0) = (0.2)\left[2z_0 - 2y_0 + e^{2x_0}\sin x_0\right] = (0.2)\left[2(-0.6) - 2(-0.4) + e^{2(0)}\sin(0)\right] = (0.2)[-0.12 + 0.8] = \cancel{0.136}\ -0.08$$

$$K_2 = h f_1\left[x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right] = (0.2) f_1\left[0 + \frac{0.2}{2}, -0.4 - \frac{0.12}{2}, -0.6 + \frac{0.136}{2}\right]$$

$$= (0.2) f_1[0.1, -0.46, -0.532] = (0.2)[-0.532] = -0.1064 \quad -0.128$$

$$l_2 = h f_2\left[x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right] = (0.2) f_2\left[0 + \frac{0.2}{2}, -0.4 - \frac{0.12}{2}, -0.6 + \frac{0.136}{2}\right]$$

$$= (0.2) f_2[0.1, -0.46, -0.532] = (0.2)\left[2(-0.532) - 2(-0.46) + e^{2(0.1)}\sin(0.1)\right]$$

$$= (0.2)[-1.064 + 0.92 + 0.1294] = -0.00292 \quad -0.0476$$

$$k_3 = hf_1\left[x_0+\frac{h}{2}, y_0+\frac{k_2}{2}, z_0+\frac{l_2}{2}\right]$$
$$= (0.2) f_1\left[0+\frac{0.2}{2}, -0.4-\frac{0.1064}{2}, -0.6-\frac{0.00292}{2}\right]$$
$$= (0.2) f_1 [0.1, -0.4532, -0.60146]$$
$$= (0.2)(-0.60146)$$
$$= \cancel{-0.1203}$$
$$-0.1248$$

$$l_3 = hf_2\left[x_0+\frac{h}{2}, y_0+\frac{k_2}{2}, z_0+\frac{l_2}{2}\right]$$
$$= (0.2) f_2\left[0+\frac{0.2}{2}, -0.4-\frac{0.1064}{2}, -0.6-\frac{0.00292}{2}\right]$$
$$= (0.2) f_2 [0.1, -0.4532, -0.6015]$$
$$= (0.2)\left[2(-0.60146)-2(-0.4532)+e^{2(0.1)}\sin(0.1)\right]$$
$$= 0.2\left[-1.20292+0.9064+0.12194\right]$$
$$= \cancel{-0.0105}\ -0.0395$$

$$k_4 = hf_1[x_0+h, y_0+k_3, z_0+l_3]$$
$$= (0.2) f_1[0+0.2, -0.4-0.1203, -0.6-0.0105]$$
$$= (0.2) f_1[0.2, -0.5203, -0.6105]$$
$$= (0.2)[-0.6105]$$
$$= \cancel{-0.1221}$$
$$-0.1279$$

$$l_4 = hf_2[x_0+h, y_0+k_3, z_0+l_3]$$
$$= (0.2) f_2[0+0.2, -0.4-0.1203, -0.6-0.0105]$$
$$= (0.2) f_2[0.2, -0.5203, -0.6105]$$
$$= (0.2)\left[2(-0.6105)-2(-0.5203)+e^{2(0.2)}\sin(0.2)\right]$$
$$= (0.2)[-1.221+1.0406+0.2964]$$
$$= 0.0825$$

$$\Delta y = \frac{1}{6}[k_1+2k_2+2k_3+k_4]$$
$$= \frac{1}{6}\left[-0.12+2(-0.1064)+2(-0.1203)+(-0.1221)\right]$$
$$= -\frac{1}{6}\left[0.12+2(0.1064)+2(0.1203)+0.1221\right]$$
$$= \cancel{-0.1159}\ -0.1256$$

$$\Delta z = \frac{1}{6}[l_1+2l_2+2l_3+l_4]$$
$$= \frac{1}{6}\left[0.136+2(-0.00292)+2(-0.0105)+0.0825\right]$$
$$= \frac{1}{6}\left[0.136-2(0.00292)-2(0.0105)+0.0825\right]$$
$$= 0.03194$$

$$y_1 = y_0+\Delta y$$
$$= -0.4-0.1159$$
i.e., $y(0.2) = \cancel{-0.5159}$
$$-0.5256$$

$$z_1 = z_0+\Delta z$$
$$= -0.6+0.3194$$
$$z(0.2) = -0.2806$$