STOKE'S THEOREM:

If $\vec{F}$ is any continuous differentiable vector function and S is a surface enclosed by a curve C then.

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,ds.$$

where $\hat{n}$ is the unit normal vector at any point of S.

③ Verify stoke's theorem from $\vec{F} = (x^2+y^2)\vec{i} - 2xy\,\vec{j}$ taken round the rectangle bounded by $x = \pm a$, $y=0$, $y=b$.

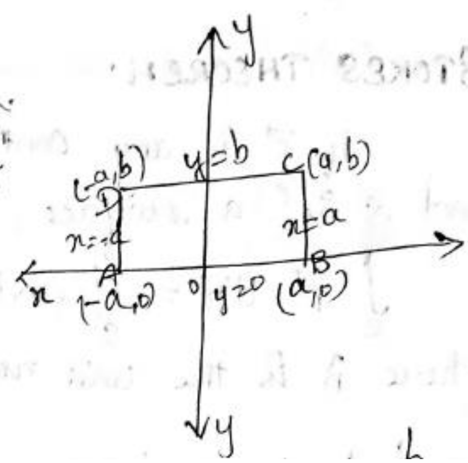
Soln. By stoke's theorem $\int_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,ds$.

Here $\vec{F} = (x^2+y^2)\vec{i} - 2xy\,\vec{j}$

Now
$$\text{curl}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x^2+y^2) & -2xy & 0 \end{vmatrix} = \vec{i}(0) - \vec{j}(0) + \vec{k}(-2y-2y)$$
$$= -4y\,\vec{k}$$

For the rectangle ABCD, $\hat{n} = \vec{k}$.

$\therefore \text{curl}\,\vec{F}\cdot\hat{n} = -4y\vec{k}\cdot\vec{k} = -4y$.



$$\iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,ds = \int_{y=0}^{b}\int_{x=-a}^{a} -4y\,dx\,dy$$

$$= \int_0^b -4y\, x \Big]_{-a}^{a} dy = \int_0^b -4y\,[a+a]\, dy = -8a\int_0^b y\, dy$$

$$= -4ab^2$$

Now $\int_C \vec{F}\cdot d\vec{r} = \int_{AB} + \int_{BC} + \int_{CD} + \int_{DA}$.

Along AB, $y=0 \Rightarrow dy=0$. x varies from $-a$ to a.

$$\int_{AB} \vec{F}\cdot d\vec{r} = \int_{-a}^{a} [(x^2+y^2)\vec{i} - 2xy\,\vec{j}]\cdot[dx\,\vec{i}+dy\,\vec{j}+dz\,\vec{k}]$$

$$= \int_{-a}^{a} (x^2+y^2)\,dx - 2xy\,dy.$$

$$= \int_{-a}^{a} x^2 dx = \frac{x^3}{3}\Big]_{-a}^{a} = \frac{2a^3}{3}.$$

Along B.C, $x=a \Rightarrow dx=0$, y varies from 0 to b

$$\int_{BC} \vec{F}\cdot d\vec{r} = \int_0^b (a^2+y^2)(0) - 2ay\,dy = -2a\int_0^b y\,dy = -2a\,\frac{y^2}{2}\Big]_0^b$$

$$= -ab^2$$

Along CD, $y=b \Rightarrow dy=0$, x varies from a to $-a$:

$$\int_{CD} \vec{F}\cdot d\vec{r} = \int_{a}^{-a} (x^2+b^2)\,dx = \left[\frac{x^3}{3}+b^2x\right]_{a}^{-a} = -\frac{a^3}{3}-ab^2-\frac{a^3}{3}-ab^2$$

$$= -2\left[ab^2+\frac{a^3}{3}\right]$$

Along DA, $x=-a \Rightarrow dx=0$, y varies from b to 0.

$$\int_{DA} \vec{F}\cdot d\vec{r} = \int_{b}^{0} (a^2+y^2)(0)+2ay\,dy = 2a\int_{b}^{0} y\,dy = \cancel{2}a\frac{y^2}{\cancel{2}}\Big]_{b}^{0}$$

$$= -ab^2.$$

$$\therefore \int_C \vec{F}\cdot d\vec{r} = \int_{AB} + \int_{BC} + \int_{CD} + \int_{DA}$$

$$= \frac{2a^3}{3} - ab^2 - 2ab^2 - \frac{2a^3}{3} - ab^2$$

$$= -4ab^2$$

$\therefore \int_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,ds$, Stoke's theorem is verified

(4) Verify Stoke's theorem for $\vec{F}=(x^2-y^2)\vec{i}+2xy\vec{j}$ in the rectangular region in the xy plane bounded by the lines $x=0$, $x=a$; $y=0$, and $y=b$. [Soln: $2ab^2$]

(5) Verify Stoke's theorem for $\vec{F} = (y-z+2)\vec{i} + (yz+4)\vec{j} - xz\vec{k}$ where S is the open surface of the cube $x=0, y=0, z=0, x=2, y=2, z=2$ above the xy-plane.

Soln: By Stoke's Theorem,

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,ds.$$

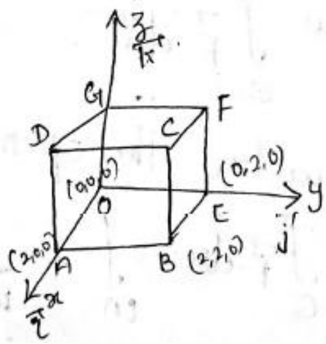
Here $\vec{F} = (y-z+2)\vec{i} + (yz+4)\vec{j} - xz\vec{k}$

$$\text{curl}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ (y-z+2) & (yz+4) & -xz \end{vmatrix} = \vec{i}(0-y) - \vec{j}(-z+1) + \vec{k}(0-1)$$

$$= -y\vec{i} + (z-1)\vec{j} - \vec{k}$$

Now $\iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,ds =$

$$\iint_{S_1} + \iint_{S_2} + \iint_{S_3} + \iint_{S_4} + \iint_{S_5}$$



Surface	$\hat{n}$	ds	face eqn.
S_1 - ABCD	$\vec{i}$	$dy\,dz$	$x=2$
S_2 - OEFG	$-\vec{i}$	$dy\,dz$	$x=0$
S_3 - BECF	$\vec{j}$	$dx\,dz$	$y=2$
S_4 - OADG	$-\vec{j}$	$dx\,dz$	$y=0$
S_5 - CDGF	$\vec{k}$	$dx\,dy$	$z=2$

$$\iint_{S_1:ABCD} \text{curl}\,\vec{F}\cdot\hat{n}\,ds = \int_0^2\int_0^2 -y\vec{i}+(z-1)\vec{j}-\vec{k}\cdot\vec{i}\;dy\,dz$$

$$= \int_0^2\int_0^2 -y\,dy\,dz = -\int_0^2 \left[\frac{y^2}{2}\right]_0^2 dz = -2\int_0^2 dz = -4$$

$$\iint_{S_2:OEFG} \text{curl}\,\vec{F}\cdot\hat{n}\,ds = \int_0^2\int_0^2 -y\vec{i}+(z-1)\vec{j}-\vec{k}\cdot(-\vec{i})\,dy\,dz$$

$$= +\int_0^2\int_0^2 y\,dy\,dz = \int_0^2 \left[\frac{y^2}{2}\right]_0^2 dz = 2\int_0^2 dz = 4$$

$$\iint_{S_3:BECF} \text{curl}\,\vec{F}\cdot\hat{n}\,ds = \int_0^2\int_0^2 (z-1)\,dx\,dz = \int_0^2 \left[zx-x\right]_0^2 dz = \int_0^2 \left[2z-2\right]dz$$

$$= \left[z^2-2z\right]_0^2 = 4-4 = 0$$

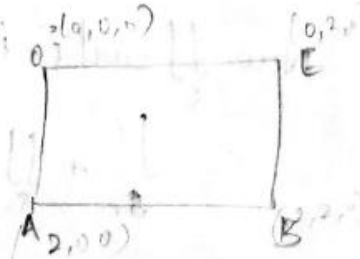
$$\iint_{S_4:OADG} \text{curl}\,\vec{F}\cdot\hat{n}\,ds = \int_0^2\int_0^2 -(z-1)\,dx\,dz = 0$$

$$\iint_{S_5\ CDGF} \text{curl}\,\vec{F}\cdot\hat{n}\,ds = \int_0^2\int_0^2 -dx\,dy = -\int_0^2 \left[x\right]_0^2 dy = -2\int_0^2 dy = -4$$

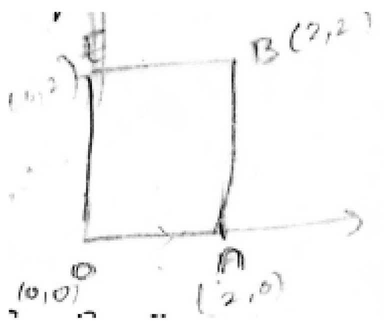
$$\therefore \iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,ds = \iint_{S_1} + \iint_{S_2} + \iint_{S_3} + \iint_{S_4} + \iint_{S_5} = -4+4+0+0-4 = -4$$

Now $\int_C \vec{F}\cdot d\vec{r}$

Consider the open surface OABE



$$\int_C \vec{F}\cdot d\vec{r} = \int_{OA} + \int_{AB} + \int_{BE} + \int_{EO}$$



Along OA, $y=0, z=0 \Rightarrow dy = dz = 0$, x varies from 0 to 2

$$\therefore \int_{OA} \vec{F}\cdot d\vec{r} = \int_0^2 (y-z+2)\,dx + (yz+4)\,dy - xz\,dz \qquad \begin{bmatrix} y=0, z=0 \\ dy=0, dz=0 \end{bmatrix}$$

$$= \int_0^2 (0-0+2)\,dx + 0$$

$$= \int_0^2 2\,dx = 4$$

Along AB, $x=2$ and $z=0 \Rightarrow dx = dz = 0$, y varies from 0 to 2

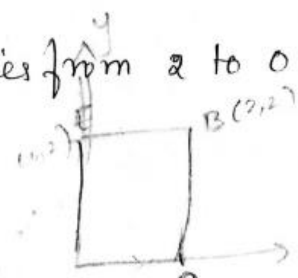
$$\int_{AB} \vec{F}\cdot d\vec{r} = \int_0^2 (y-z+2)\,dx + (yz+4)\,dy - xz\,dz \qquad \begin{bmatrix} z=0;\ x=2 \\ dz = dx = 0 \end{bmatrix}$$

$$= \int_0^2 4\,dy = 8$$

Along BE, $y=2, z=0 \Rightarrow dy = dz = 0$, x varies from 2 to 0

$$\int_{BE} \vec{F}\cdot d\vec{r} = \int_2^0 4\,dx = -8$$

Along EO, $x=0, z=0 \Rightarrow dx = dz = 0$, y varies from 2 to 0



$$\int_{EO} \vec{F}\cdot d\vec{r} = \int_2^0 4\,dy = -8$$

$$\therefore \int_C \vec{F}\cdot d\vec{r} = \int_{OA} + \int_{AB} + \int_{BE} + \int_{EO} = 4 + 8 - 8 - 8 = -4$$

$$\therefore \int_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,ds.$$ Stokes theorem verified.