GREEN'S THEOREM IN THE PLANE:

If R is a closed region of the xy plane bounded by a simple closed curve C and if M and N are continuous functions of x and y having continuous derivatives in R then

$$\int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

where C is a curve described traversed in the anticlockwise direction.

Type-I

(1) Evaluate by Green's theorem $\int (xy+x^2)dx + (x^2+y^2)dy$ where C is the square formed by $x=-1$, $x=1$, $y=-1$, $y=1$

Soln: Let R be the region enclosed by C. Green's theorem is

$$\int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

Here $M = xy + x^2 \Rightarrow \frac{\partial M}{\partial y} = x$

$N = x^2 + y^2 \Rightarrow \frac{\partial N}{\partial x} = 2x$

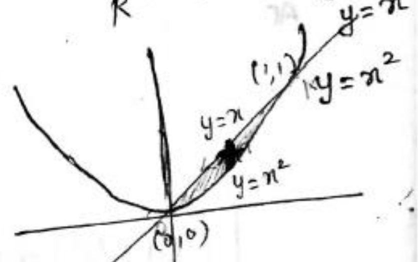
$$\int_C (xy+x^2)dx + (x^2+y^2)dy = \iint_R (2x-x)\,dx\,dy$$

$$= \int_{-1}^{1}\int_{-1}^{1} x\,dx\,dy = \int_{-1}^{1} \left[\frac{x^2}{2}\right]_{-1}^{1} dy = 0$$

Type-II

4) Verify Green's theorem in the plane for $\int_C (xy+y^2)\,dx + x^2\,dy$ where C is the closed curve of the region bounded by $y=x$ and $y=x^2$.

Soln: By Green's theorem $\int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$



Here $M = xy+y^2$, $N = x^2$.

$\Rightarrow \frac{\partial M}{\partial y} = x+2y$, $\frac{\partial N}{\partial x} = 2x$.

$\therefore \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \iint (2x - x - 2y)\,dy\,dx$

Gn. $y=x^2$ & $y=x$

$\Rightarrow x^2 = x$.

$\Rightarrow x^2 - x = 0$

$\Rightarrow x(x-1) = 0$

$\Rightarrow x=0, x=1$

$y=x$; $y=0, y=1$

$(0,0)$ & $(1,1)$

$$= \int_0^1 \int_{x^2}^{x} (x - 2y)\,dy\,dx$$

$$= \int_0^1 \left[xy - y^2\right]_{x^2}^{x} dx$$

$$= \int_0^1 x^2 - x^2 - \left[x^3 - x^4\right] dx$$

$$= -\int_0^1 (x^3 - x^4)\,dx$$

$$= -\left[\frac{x^4}{4} - \frac{x^5}{5}\right]_0^1 = -\left[\frac{1}{4} - \frac{1}{5}\right] = -\frac{1}{20} \quad \text{— (1)}$$

Along OA, $y = x^2$

Along $y = x^2$, x varies from 0 to 1

$dy = 2x\,dx$.

$$\int_{OA} (xy + y^2)\,dx + x^2\,dy = \int_0^1 (x^3 + x^4)\,dx + x^2(2x\,dx)$$

$$= \int_0^1 (3x^3 + x^4)\,dx$$

$$= 3\frac{x^4}{4} + \frac{x^5}{5}\Big]_0^1 = \frac{3}{4} + \frac{1}{5} = \frac{19}{20}$$

Along AO, $y = x$.

Along $y = x$, x varies from 1 to 0

$dy = dx$.

$$\int_{AO} (xy + y^2)\,dx + x^2\,dy = \int_1^0 (x^2 + x^2)\,dx + x^2\,dx = \int_1^0 3x^2\,dx$$

$$= x^3\Big]_1^0 = -1$$

$$\therefore \int_C (xy + y^2)\,dx + x^2\,dy = \int_{OA} + \int_{AO}$$

$$= \frac{19}{20} - 1 = -\frac{1}{20} \quad \text{—(2)}$$

Hence Green's theorem is verified by comparing (1) & (2)

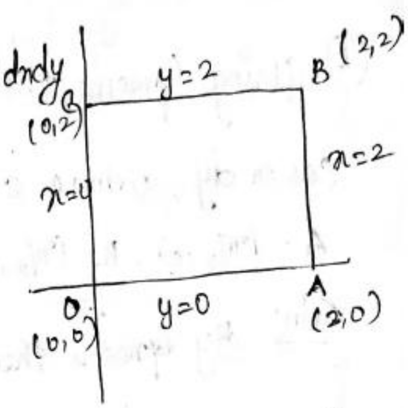
(5) Verify Green's theorem in the plane for $\int_C (x^2 - xy^3)\,dx + (y^2 - 2xy)\,dy$ where C is the square with vertices $(0,0)$, $(2,0)$, $(2,2)$, $(0,2)$.

Soln: By Green's theorem, $\int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$.

Here $M = x^2 - xy^3 \Rightarrow \frac{\partial M}{\partial y} = -3xy^2$

$N = y^2 - 2xy \Rightarrow \frac{\partial N}{\partial x} = -2y$.

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \int_0^2\int_0^2 (-2y + 3xy^2)\,dx\,dy$$



$$= \int_0^2 \left[-2xy + 3\frac{x^2}{2}y^2\right]_0^2 dy$$

$$= \int_0^2 (-4y + 6y^2)\,dy$$

$$= \left[-4\frac{y^2}{2} + 6\frac{y^3}{3}\right]_0^2$$

$$= -8 + 16 = 8$$

Along OA, $y = 0 \Rightarrow dy = 0$, x varies from 0 to 2.

$$\int_{OA} (x^2 - xy^3)\,dx + (y^2 - 2xy)\,dy = \int_0^2 x^2\,dx = \left[\frac{x^3}{3}\right]_0^2 = \frac{8}{3}.$$

Along AB, $x=2 \Rightarrow dx=0$, y varies from 0 to 2.

$$\int_{AB} (x^2-xy^3)dx+(y^2-2xy)dy = \int_0^2 (y^2-4y)\,dy$$

$(4-2y^3)+(y^2-4y)$

$$= \left.\frac{y^3}{3} - 4\frac{y^2}{2}\right]_0^2 = \frac{8}{3} - 8 = -\frac{16}{3}$$

Along BC, $y=2 \Rightarrow dy=0$. x varies from 2 to 0

$$\int_{BC} (x^2-xy^3)dx+(y^2-2xy)dy = \int_2^0 (x^2-8x)dx$$

$$= \left[\frac{x^3}{3} - 8\frac{x^2}{2}\right]_2^0 = -\left[\frac{8}{3}-16\right] = \frac{40}{3}$$

Along CO, $x=0 \Rightarrow dx=0$, y varies from 2 to 0.

$$\int_{CO} (x^2-xy^3)dx+(y^2-2xy)dy = \int_2^0 y^2\,dy = \left.\frac{y^3}{3}\right]_2^0 = -\frac{8}{3}.$$

$$\therefore \int_C (x^2-xy^3)dx+(y^2-2xy)dy = \int_{OA} + \int_{AB} + \int_{BC} + \int_{CO}$$

$$= \frac{8}{3} - \frac{16}{3} + \frac{40}{3} - \frac{8}{3} = 8$$

7) Apply Green's theorem in the plane to evaluate $\int_C (3x^2-8y^2)\,dx + (4y-6xy)\,dy$ where C is the boundary of the region defined by $x=0$, $y=0$ & $x+y=1$. Soln: $5/3$

Area of the region R enclosed by the curve C is $\frac{1}{2}\int_C x\,dy - y\,dx$.

(9) Evaluate $\int_C (2xy-x^2)\,dx + (x+y^2)\,dy$ using Green's theorem where C is the closed curve formed by $y=x^2$ & $y^2=x$.

Soln: $1/30$.