Lecture 35

NAVIER-STOKES EQUATIONS: FRACTIONAL STEP METHODS

35.1 FRACTIONAL STEP METHODS

Fractional step methods are essentially approximate factorization methods. In the case explicit time integration, say, using explicit Euler method, the discretized Navier-Stokes equation can be represented as:

$$v_i^{n+1} = v_i^n + \Delta t\left(C_i + D_i + P_i\right) \tag{35.1}$$

which can be split into a three steps method

$$v_i^* = v_i^n + \left(C_i\right)\Delta t \tag{35.2}$$

$$v_i^{**} = v_i^* + \left(D_i\right)\Delta t \tag{35.3}$$

$$v_i^{n+1} = v_i^{**} + \left(P_i\right)\Delta t \tag{35.4}$$

In the third step P_i is the gradient of a quantity that obeys a Poisson equation. Actual form of the algorithm would depend on the choice of discretization methods, handling of convective terms etc.

Next, let us consider a fractional step method based on Crank-Nicolson method. In the first step, the velocity is advanced using pressure from previous time step, whereas rest of the terms are handled as in standard Crank-Nicolson method, i.e.

$$\frac{\left(\rho v_i^*\right) - \left(\rho v_i^n\right)}{\Delta t} = \frac{1}{2}\left[H\left(v_i^n\right) + H\left(v_i^*\right)\right] - \frac{\delta p^n}{\delta x_i} \tag{35.5}$$

System of equation (35.5) must be solved for v_i^*. In the second step, half the old pressure gradient is removed leading to

$$\frac{\rho v_i^{**} - \rho v_i^*}{\Delta t} = \frac{1}{2}\left(\frac{\delta p^n}{\delta x_i}\right) \tag{35.6}$$

The final velocity at the next time level requires the gradient of pressure at new time level. Hence,

$$\frac{\left(\rho v_i\right)^{n+1} - \left(\rho v_i\right)^{**}}{\Delta t} = -\frac{1}{2}\frac{\delta p^{n+1}}{\delta x_i} \tag{35.7}$$

New velocity from (35.7) must satisfy the continuity equation, which leads to a Poisson equation for the new pressure

$$\frac{\delta}{\delta x_i}\left(\frac{\delta p^{n+1}}{\delta x_i}\right)=\frac{2}{\Delta t}\frac{\delta\left(\rho v_i^{**}\right)}{\delta x_i} \tag{35.8}$$

In algorithmic from, this method can be expressed by the following steps at the new time level t_{n+1}:

- Solve the system of equations (35.5) to obtain v_i^*
- Compute v_i^{**} using Eq. (35.6)
- Solve the pressure Poisson equation (35.8) to get p^{n+1}
- Obtain v_i^{n+1} from Eq. (35.7)

There are wide varieties of fractional step method due to vast choice of approaches for time and space discretization. However, all of them are based on the principle described above.

FURTHER READING

Ferziger, J. H. And Perić, M. (2003). *Computational Methods for Fluid Dynamics*. Springer.

Versteeg, H. K. and Malalasekera, W. M. G. (2007). *Introduction to Computational Fluid Dynamics: The Finite Volume Method*. Second Edition (Indian Reprint) Pearson Education.