Generic form of Governing flow eqs with strong conservative form in transformed space.

Consider a 2d conservative form of governing flow eqs (Generic form) in physical plane with no source term is given as

$$\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} = 0 \longrightarrow (47)$$

To transform eq. 47 in (ξ, η) space as below

$$\boxed{\frac{\partial U_1}{\partial t} + \frac{\partial F_1}{\partial \xi} + \frac{\partial G_1}{\partial \eta} = 0} \longrightarrow (48)$$

where F_1 & G_1 should be suitable combination of original flux vectors F & G. ~~If so it is able to retain our transformed space~~

Transform the spatial variables in eq 47 as per derivative transformation by eq. (4) & (5)

$$\frac{\partial U}{\partial t} + \frac{\partial F}{\partial \xi}\left(\frac{\partial \xi}{\partial x}\right) + \frac{\partial F}{\partial \eta}\left(\frac{\partial \eta}{\partial x}\right) + \frac{\partial G}{\partial \xi}\left(\frac{\partial \xi}{\partial y}\right) + \frac{\partial G}{\partial \eta}\left(\frac{\partial \eta}{\partial y}\right) = 0 \longrightarrow (49)$$

Multiply eq. (49) by Jacobian J

$$J\frac{\partial U}{\partial t} = J\left(\frac{\partial F}{\partial \xi}\right)\left(\frac{\partial \xi}{\partial x}\right) + J\left(\frac{\partial F}{\partial \eta}\right)\left(\frac{\partial \eta}{\partial x}\right) + J\left(\frac{\partial G}{\partial \xi}\right)\left(\frac{\partial \xi}{\partial y}\right) + J\left(\frac{\partial G}{\partial \eta}\right)\left(\frac{\partial \eta}{\partial y}\right) = 0 \longrightarrow (50)$$

Consider an simple derivative expansion of term

$$\frac{\partial\left[JF\left(\partial \xi / \partial x\right)\right]}{\partial \xi} = J\left(\frac{\partial \xi}{\partial x}\right)\left(\frac{\partial F}{\partial \xi}\right) + \cancel{F\left(\frac{\partial \xi}{\partial x}\right)\left(\frac{\partial}{}\right)} \; F\frac{\partial}{\partial \xi}\left(J\frac{\partial \xi}{\partial x}\right) \longrightarrow 51$$

Re-arranging eq 51 we have

$$J\left(\frac{\partial F}{\partial \xi}\right)\left(\frac{\partial \xi}{\partial x}\right) = \frac{\partial\left[JF(\partial\xi/\partial x)\right]}{\partial \xi} - F\frac{\partial}{\partial \xi}\left(J\frac{\partial \xi}{\partial x}\right) \rightarrow 52$$

Similarly taking η derivative of $JF(\partial\eta/\partial\eta)$

$$J\left(\frac{\partial F}{\partial \eta}\right)\left(\frac{\partial \eta}{\partial x}\right) = \frac{\partial\left[JF(\partial\eta/\partial x)\right]}{\partial \eta} - F\frac{\partial}{\partial \eta}\left(J\frac{\partial \eta}{\partial x}\right) \rightarrow 53$$

Similarly the terms $JG(\partial\eta/\partial y)$ & $JG\left(\frac{\partial\xi}{\partial y}\right)$ can be expanded & re arranged

$$J\left(\frac{\partial G}{\partial \xi}\right)\left(\frac{\partial \xi}{\partial y}\right) = \frac{\partial\left[JG(\partial\xi/\partial y)\right]}{\partial \xi} - G\frac{\partial}{\partial \xi}\left(J\frac{\partial \xi}{\partial y}\right) \rightarrow 54$$

$$J\left(\frac{\partial G}{\partial \eta}\right)\left(\frac{\partial \eta}{\partial y}\right) = \frac{\partial\left[JG(\partial\eta/\partial y)\right]}{\partial \eta} - G\frac{\partial}{\partial \eta}\left(J\frac{\partial \eta}{\partial y}\right) \rightarrow 55$$

Substitute eq 52, 53, 54 & 55 in eq 50 & re factoring we have.

$$J\frac{\partial U}{\partial t} + \frac{\partial}{\partial \xi}\left(JF\frac{\partial \xi}{\partial x} + JG\frac{\partial \xi}{\partial y}\right) + \frac{\partial}{\partial \eta}\left(JF\frac{\partial \eta}{\partial x} + JG\frac{\partial \eta}{\partial y}\right)$$

$$-F\left[\frac{\partial}{\partial \xi}\left(J\frac{\partial \xi}{\partial x}\right) + \frac{\partial}{\partial \eta}\left(J\frac{\partial \eta}{\partial x}\right)\right] - G\left[\frac{\partial}{\partial \xi}\left(J\frac{\partial \xi}{\partial y}\right) + \frac{\partial}{\partial \eta}\left(J\frac{\partial \eta}{\partial y}\right)\right]$$

$$= 0 \longrightarrow 56$$

Last two terms in eq 56 are zero. as follows

$$\left[\frac{\partial}{\partial \xi}\left(J\frac{\partial \xi}{\partial x}\right) + \frac{\partial}{\partial \eta}\left(J\frac{\partial \eta}{\partial x}\right)\right] = \frac{\partial}{\partial \xi}\left(\frac{\partial y}{\partial \eta}\right) - \frac{\partial}{\partial \eta}\left(\frac{\partial y}{\partial \xi}\right)$$

$$= \frac{\partial^2 y}{\partial \xi \partial \eta} - \frac{\partial^2 y}{\partial \eta \partial \xi} = 0$$

$$\frac{\partial}{\partial \xi}\left(J\frac{\partial \xi}{\partial y}\right) + \frac{\partial}{\partial \eta}\left(J\frac{\partial \eta}{\partial y}\right) = \frac{\partial}{\partial \xi}\left(-\frac{\partial x}{\partial \eta}\right) + \frac{\partial}{\partial \eta}\left(\frac{\partial x}{\partial \xi}\right)$$

$$-\frac{\partial^2 x}{\partial \xi \partial \eta} + \frac{\partial^2 x}{\partial \eta \partial \xi} = 0$$

Thus eq 56

$$\frac{\partial U_1}{\partial t} + \frac{\partial F_1}{\partial \xi} + \frac{\partial G_1}{\partial \eta} = 0 \longrightarrow 57$$

$$U_1 = JU \longrightarrow 58$$

$$F_1 = JF\frac{\partial \xi}{\partial x} + JG\frac{\partial \xi}{\partial y} \longrightarrow 59$$

$$G_1 = JG\frac{\partial \eta}{\partial x} + JG\frac{\partial \eta}{\partial y} \longrightarrow 60$$

As per eq 57 generic form of governing eq is written in strong conservative form in transformed space.

F_1 & G_1 are combination of flux vectors F & G where combinations involve Jacobian J & direct metrices.