Unit-3

Mathematical Behaviour of Partial differential Equations; The Impact of CFD.

Pre-requisite:-

Quasi linear system of Equations

Higher order derivatives occur linearly; there are no products & exponentials of highest order derivatives- they occur by themselves, multiplied by co-effecients which are functions of ~~linear~~ dependent variable themselves. Such a system of equations is known as quasi- linear system. These eqs are similar to governing PD Eqs of fluid dynamics as derived earlier (Ex;- Euler's eq)

Classification of PDE (Partial differential Eqs) → by.

1) Cramer Method

Consider a system of quasi- linear equations given as below

$$a_1 \frac{\partial u}{\partial x} + b_1 \frac{\partial u}{\partial y} + c_1 \frac{\partial v}{\partial x} + d_1 \frac{\partial v}{\partial y} = f_1 \longrightarrow (1)$$

$$a_2 \frac{\partial u}{\partial x} + b_2 \frac{\partial u}{\partial y} + c_2 \frac{\partial v}{\partial x} + d_2 \frac{\partial v}{\partial y} = f_2 \longrightarrow (2)$$

Here u & v are dependent variables, functions of x & y.

The coeffecients $a_1, a_2, b_1, b_2, c_1, c_2$ d_1 & d_2 f_1 & f_2 can be functions of x, y, u & v.

→ u & v are continous function of x & y

→ u & v represent continous velocity field throughout xy space.

(2)

At a given point in x. y space, there is a unique ~~way~~ value of u & v respectively. More over the derivatives of u & v $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$, $\frac{\partial v}{\partial y}$ are finite values at given point.

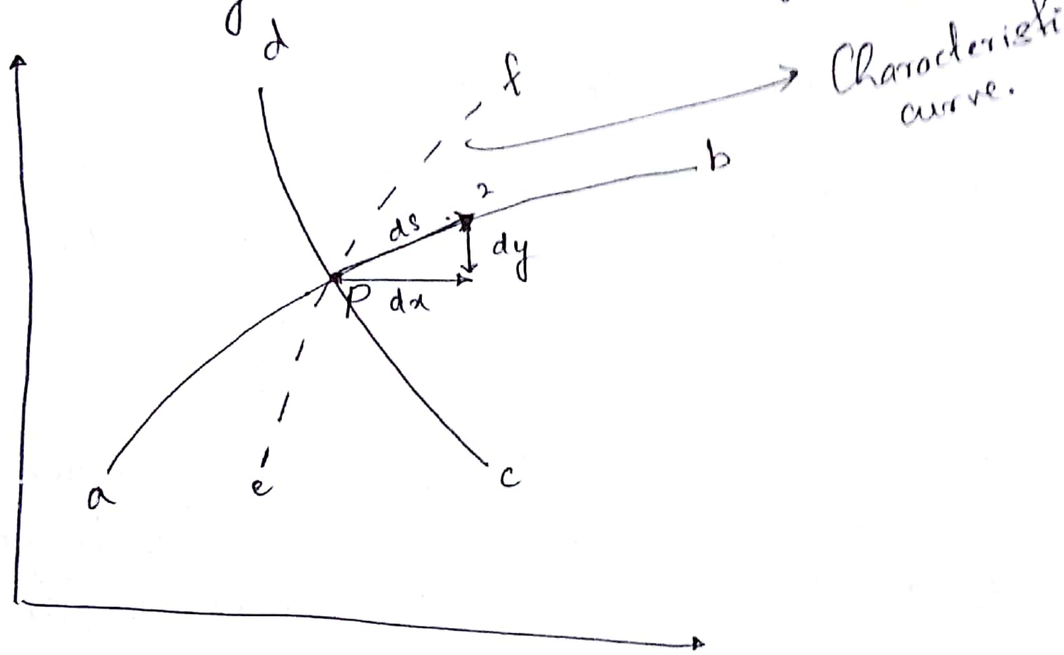

Fig 1:- Illustration of a characteristic curve

Consider a point as given in Fig 1 above, in the xy plane. Let us seek the lines through this points along which derivatives of u & v are indeterminant & across which can also be discontinous. These lines which we are seeking is known as characteristic lines.

To find such lines recall u & v as continous functions of x & y & we have total differentials as.

$$du = \frac{\partial u}{\partial x} \cdot dx + \frac{\partial u}{\partial y} \cdot dy \rightarrow (3)$$

$$dv = \frac{\partial v}{\partial x} \cdot dx + \frac{\partial v}{\partial y} \cdot dy \rightarrow (4)$$

Eqs ①, ② 3 & ④ constitute a system of four linear eqs with four unknows ($\partial u/\partial x$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$, $\frac{\partial v}{\partial y}$). The eqs can be written in matrix form as below.

$$\underbrace{\begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ dx & dy & 0 & 0 \\ 0 & 0 & dx & dy \end{bmatrix}}_{[A]} \begin{bmatrix} \partial u/\partial x \\ \partial u/\partial y \\ \partial v/\partial x \\ \partial v/\partial y \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ du \\ dv \end{bmatrix} \rightarrow (5)$$

[A]
Co-effecient matrix.

To solve eq ⑤ for unknow $\partial u/\partial x$ using cramer's rule To do this we define matrix [B] as matrix [A] with its first column replaced by column vector on RHS of eq. ⑤

$$[B] \equiv \begin{bmatrix} f_1 & b_1 & c_1 & d_1 \\ f_2 & b_2 & c_2 & d_2 \\ du & dy & 0 & 0 \\ dv & 0 & dx & dy \end{bmatrix} \longrightarrow (6)$$

The determinants of matrix [A] & [B] are ~~given~~ denoted by |A| & |B|. By cramers rule the solution for $\partial u/\partial x$ is given as

$$\frac{\partial u}{\partial x} = \frac{|B|}{|A|} \rightarrow (7)$$

To get actual number for $\frac{\partial u}{\partial x}$ from eq. ⑦, we have to establish values for du, dy, dv, dx. For this let us imagine curve ab drawn in an arbitrary direction through point P. Following curve ab till point 2 as shown in fig 1. The distance travelled from point P to point 2 is given as ds.

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The change in x associated ~~in~~ with moving from Point P to point Q is $dx = -(x_p - x_Q)$ or $x_Q - x_p$ & the associated change in y-direction is $dy = y_Q - y_p$. These are values of dx & dy. Similarly the value of u & v at point P & Q do differ, they have changed by amounts $du = u_Q - u_p$ & $dv = v_Q - v_p$. Insertive values of dx, dy, du & dv in eq. 5 & eq. 6 we can obtain soln for $\partial v/\partial x$. in the limiting cas as dx & dy goes to zero.

Now, drawing another arbitrary curve cd, we can repeat the same process as above for curve ab, i.e to move through infinitesimally small distance ds. from point P along curve cd. Then in similar manner get values of dx, dy, du, dv. These values are different as we are moving in diffent direction.

However when the values are inserted into eq 5 & 6 in the limiting case as dx & dy go to zero, the same value of $\partial v/\partial x$ is obtained as earlier.

This has to be case as $\partial v/\partial x$ is fixed at point P. However there is an exception to above formalism. When $|A| = 0$, let us assume ef as such a direction from fig 1. Then in eq 7 denominator is zero. Then $\partial v/\partial x$ is an indeterminent value, when we choose this direction

Now let us consider ~~any~~ a point P in xy plane through which we can seek lines of directions along which u & v are indeterminant & across which may be even discontinous.

Thus a characteristic lines such as ef for which u, & v are indeterminent could be found by setting $|A| = 0 \rightarrow$ (8)

These lines are independent of whether we are solving eq. (5) for $\partial u/\partial x$, $\partial u/\partial y$, $\frac{\partial v}{\partial x}$ & $\frac{\partial v}{\partial y}$.

As per cramer rule for all above 4 cases, the denominator i.e. $|A|$ is same.

To calculate eq. of the characteristic (lines) ~~eq.~~ of curves & to find slope of the curve at point P. It could be done using eq. (8)

$$|A| = 0$$

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ dx & dy & 0 & 0 \\ 0 & 0 & dx & dy \end{vmatrix} = 0$$

Expanding the determinant.

$$(a_1 c_2 - a_2 c_1)(dy)^2 - (a_1 d_2 - a_2 d_1 + b_1 c_2 - b_2 c_1)\,dx\,dy + (b_1 d_2 - b_2 d_1)(dx)^2 \longrightarrow \text{eq. } (9)$$

Divide eq. 9 by $(dx)^2$

$$(a_1 c_2 - a_2 c_1)\left(\frac{dy}{dx}\right)^2 - (a_1 d_2 - a_2 d_1 + b_1 c_2 - b_2 c_1)\left(\frac{dy}{dx}\right) + (b_1 d_2 - b_2 d_1) \rightarrow (10)$$

Eq. (10) is quadratic eq for dy/dx

It will give slope of characteristic curve.

Let $a_0 = (a_1 c_2 - a_2 c_1)$; $b = -(a_1 d_2 - a_2 d_1 + b_1 c_2 - b_2 c_1)$;

$c = (b_1 d_2 - b_2 d_1)$

Then eq. (10) can be written as

$$a\left(\frac{dy}{dx}\right)^2 + b\left(\frac{dy}{dx}\right) + c = 0 \rightarrow (11)$$

Eq. (11) can be integrated to give $y = y(x)$ which is eq of characteristic curve in xy plane. To find slopes of characteristic curves through point P.

$$\frac{dy}{dx} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \rightarrow (12)$$

Here.
$D = b^2 - 4ac \rightarrow (13)$

Eq (12) gives direction of characteristic curve through a given point P in xy plane. These lines have different nature depending on value of D given by eq. (13)

If $D > 0$ — Then two real & distinct characteristic exists through each point in x, y plane. The system of eqs is given by eqs 1 & 2 is called hyperbolic.

If $D = 0$ — Here the system of eqs (1) & (2) are known as parabolic.

If $D < 0$ — Characteristic lines are imaginary, then the system of eqs given by (1) & (2) are elliptic.

General equation of a conic section for analytical geometry is

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

where if
$b^2 - 4ac > 0 \rightarrow$ conic - hyperbolic
$b^2 - 4ac = 0 \rightarrow$ conic - parabola
$b^2 - 4ac < 0 \rightarrow$ conic - ellipse.

From eqn (7) if only $|A| = 0$ $\frac{\partial v}{\partial x}$ → will be infinite, however the definition of characteristic curve states that $\partial v/\partial x$ is indeterminent not an infinite. Thus $|B|$ should also be equal to be zero. for $\partial v/\partial x$ to be indeterminant.

$$\frac{\partial v}{\partial x} = \frac{|B|}{|A|} = \frac{0}{0} \longrightarrow (14)$$

$$|B| = \begin{vmatrix} f_1 & b_1 & c_1 & d_1 \\ f_2 & b_2 & c_2 & d_2 \\ du & dy & 0 & 0 \\ dv & 0 & dx & dy \end{vmatrix} = 0 \longrightarrow (15)$$

Eq (14) gives $\frac{\partial v}{\partial x}$ in indeterminant form which can have a finite value.

Solving eq. (15) we get a ordinary differential eq in terms of du & dv, where dx & dy are restricted to hold along a characteristic curve. This eqs for dependent variables u & v which comes from eq. (15) is compatability eq. It is an eq involving unknown dependent variables holding only along characteristic line.

Advantage is it is one less dimension than original PD eqs (1) & (2) - & that one dimension is along characteristic direction.

The leads to solution technique for original system of eqs given (1) & (2), wherein characteristic eq are formed in xy space & the compatibility eq is solved along the characterist