Higher order difference Quotients

If we are dealing with Inviscid flows, which are governed by Euler's equations, the highest order derivatives which appear in Euler's eqs are first order partial derivatives. Hence finite order differences expressed by eqs 5, 8, 10 are enough to handle them.

But if we are dealing with viscous flow, governed by N-S eqs in which higher order partial derivative terms do occur, then we require second order finite differences.

The second order finite differences are obtained from Taylor series analysis as follows

Adding eq (1) & eq. (7)

$$u_{i+1,j} + u_{i-1,j} = 2u_{i,j} + \left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j}(\Delta x)^2 + \left(\frac{\partial^4 u}{\partial x^4}\right)\frac{(\Delta x)^4}{12} + \cdots \quad (12)$$

Solving for $\left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j}$ from eq. 12 we have

$$\boxed{\left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{(\Delta x)^2} + O(\Delta x)^2} \quad (13)$$

From eq. (13) the first term on RHS is central finite difference for second derivative w.r.t to x evaluated at grid point (i,j), from remaining order of magnitude term, we see that this central difference is of second order accuracy

An analogous expression could be obtained for second derivative w.r.t to y

$$\left(\frac{\partial^2 u}{\partial y^2}\right) = \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{(\Delta y)^2} + O(\Delta y)^2 \rightarrow (14)$$

Eq (13) & (14) are examples of second order central differences.

To get a mixed derivative $\frac{\partial^2 u}{\partial x \partial y}$

Differentiating eq (1) w.r.t to y we have.

$$\left(\frac{\partial u}{\partial y}\right)_{i+1,j} = \left(\frac{\partial u}{\partial y}\right)_{i,j} + \left(\frac{\partial^2 u}{\partial x \partial y}\right)_{i,j} \Delta x + \left(\frac{\partial^3 u}{\partial x^2 \partial y}\right)\frac{(\Delta x)^2}{2!} + \left(\frac{\partial^4 u}{\partial x^3 \partial y}\right)\frac{(\Delta x)^3}{6} + \ldots \rightarrow (15)$$

Differentiating eq. (7) w.r.t to y we have.

$$\left(\frac{\partial u}{\partial y}\right)_{i-1,j} = \left(\frac{\partial u}{\partial y}\right)_{i,j} - \left(\frac{\partial^2 u}{\partial x \partial y}\right)_{i,j} \Delta x + \left(\frac{\partial^3 u}{\partial x^2 \partial y}\right)\frac{(\Delta x)^2}{2!} - \left(\frac{\partial^4 u}{\partial x^3 \partial y}\right)\frac{(\Delta x)^3}{6} + \ldots \rightarrow (16)$$

Sub eq (16) - eq. (15)

$$\left(\frac{\partial u}{\partial y}\right)_{i+1,j} - \left(\frac{\partial u}{\partial y}\right)_{i-1,j} = 2\left(\frac{\partial^2 u}{\partial x \partial y}\right)_{i,j} \Delta x + \left(\frac{\partial^4 u}{\partial x^3 \partial y}\right)\frac{(\Delta x)^3}{6} \rightarrow (17)$$

Solving for $\left(\frac{\partial^2 u}{\partial x \partial y}\right)_{i,j}$ from eq. (16) we have

$$\left(\frac{\partial^2 u}{\partial x \partial y}\right)_{i,j} = \frac{\left(\frac{\partial u}{\partial y}\right)_{i+1,j} - \left(\frac{\partial u}{\partial y}\right)_{i-1,j}}{2\Delta x} + O(\Delta y)^2 \rightarrow (18)$$

The first term on R.H.S involves $\left(\frac{\partial u}{\partial y}\right)_{i+1,j}$ & then at grid point $(i-1, j)$. We know that $\frac{\partial u}{\partial y}$ at both these points could be replaced with a second order difference ~~patte~~, using appropriate grid points centered first at $(i+1, j)$ & then on $(i-1, j)$, as shown below.

$$\left(\frac{\partial u}{\partial y}\right)_{i+1,j} = \frac{u_{i+1,j+1} - u_{i+1,j-1}}{2\Delta y} + O(\Delta y)^2 \rightarrow 19(a)$$

$$\left(\frac{\partial u}{\partial y}\right)_{i-1,j} = \frac{u_{i-1,j+1} - u_{i-1,j-1}}{2\Delta y} + O(\Delta y)^2 \longrightarrow 19b$$

Thus substituting 19a & 19b in eq. (18) we have.

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{u_{i+1,j+1} - u_{i-1,j+1} - u_{i+1,j-1} + u_{i-1,j-1}}{4\Delta x \Delta y} + O[(\Delta x)^2, (\Delta y)^2] \rightarrow (20)$$

In the truncation error the lowest order terms are of order $(\Delta x)^2$, $(\Delta y)^2$ - thus giving second order central difference for mixed derivatives.

After this read

From. Pg. No → 134-136 (fills pro's & Con's read in text book given provided With notes.