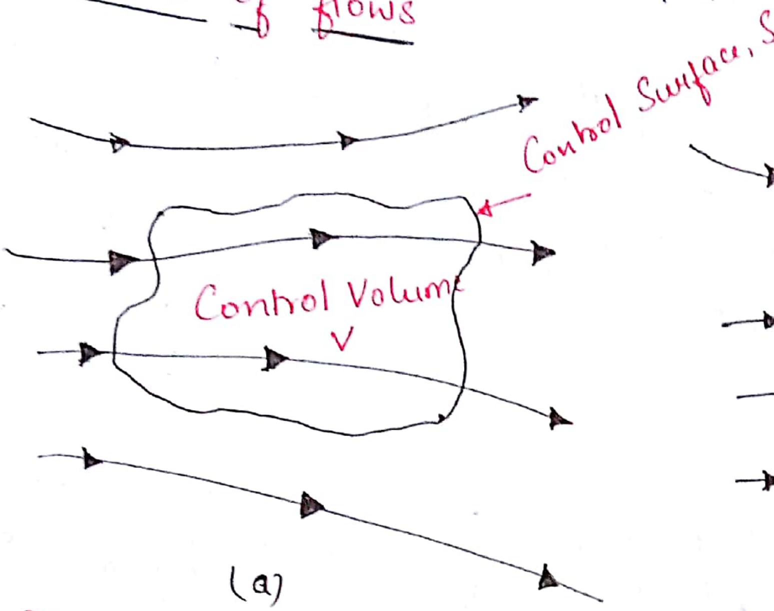
Models of flows

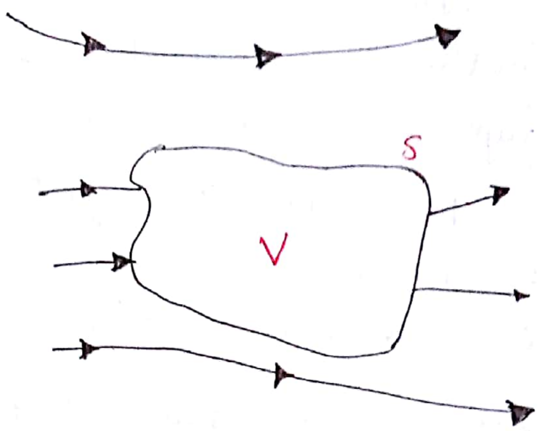
Unit - 1 & 2.

(1)



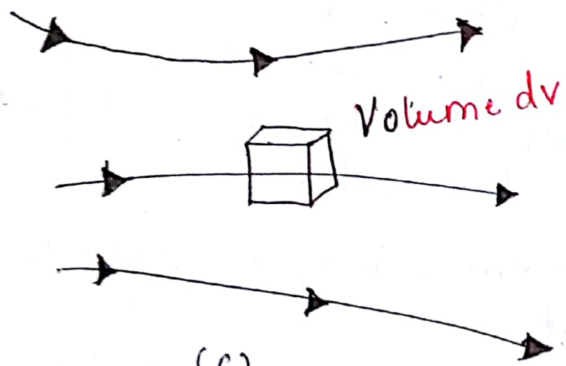
(a)

Finite control volume fixed in space with the fluid moving through it



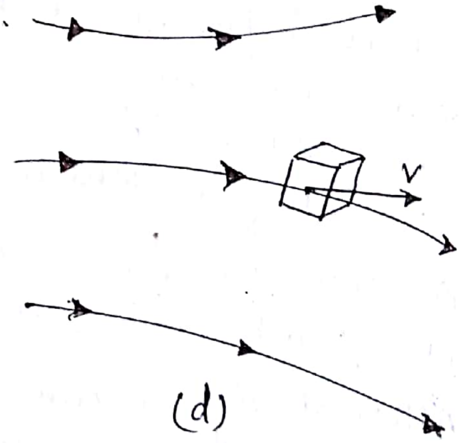
(b)

Finite control volume moving with the fluid such that the same fluid particles are always in the same control volume



(c)

Infinitesimal fluid element fixed in space with the fluid moving through it



(d)

Infinitesimal fluid element moving along a streamline with velocity v equal to local flow velocity at each point.

→ A solid body is easy to grab as we can clearly see it but where as a squishy substance 'fluid' it is hard to grab it.

→ Also a solid body having translational motion, the velocity of each part of body is same whereas when we conside a fluid in motion, the velocity may be different at each location of the fluid.

Now to derive basic equations of fluid motions (i.e. Continuity, Momentum & Energy eq) we use fundamental physical principles from law of physics, ~~that~~ ~~is~~. i.e.

a) Mass is conserved

b) F = Ma

c) Energy is conserved

For Applying these physical principles we need to convert a Continnum fluid into & Models of fluid flow, by applying we can extract mathematical equations which embody such physical principles.

Models of ~~fluids~~ flow represented as a Control Volume

→ As seen in figures above (fig a, b, c d) a general flow field is represented by streamlines. A close volume is drawn with finite region of flow. This defines a control volume V ; a control surface S is defined as a close surface which bounds the volume.

a) → As seen in fig (a) control volume may be fixed in space with fluid moving through it.

b) → Alternatively we can see in fig (b) the control volume may be moving with the fluid such that same fluid parti-cles are always inside it.

In either of above case (a) & case (b) CV is resonably large, finite region of flow. The fundamental physical principles are applied to fluid inside the CV & to fluid crossing the contol surface (if CV is fixed)

Thus looking only at CV model we limit our attention to fluid in the finite region of volume itself.

The fluid flow equations that we directly obtain by applying fundamental physical principles to a finite CV are integral in form.

By manipulation we can convert these integral equation into partial differential equations.

The equations obtained from finite CV fixed in space in either partial differential or integral form are conservative in nature.

The equations obtained from finite CV moving with the fluid in either integral or partial differential form are non-conservative in nature.

Models of fluid represented as Infinitesimal fluid element.

A general flow field is represented with streamlines.

Imagine a infinitesimal small fluid element with a differentia volume dV.

The fluid element is infinitesimal in the same sense as differential calculus, however it is large enough to contain a huge no. of molecules so that it is considered as a continuum.

The fluid element can be fixed in space as showed in fig C.

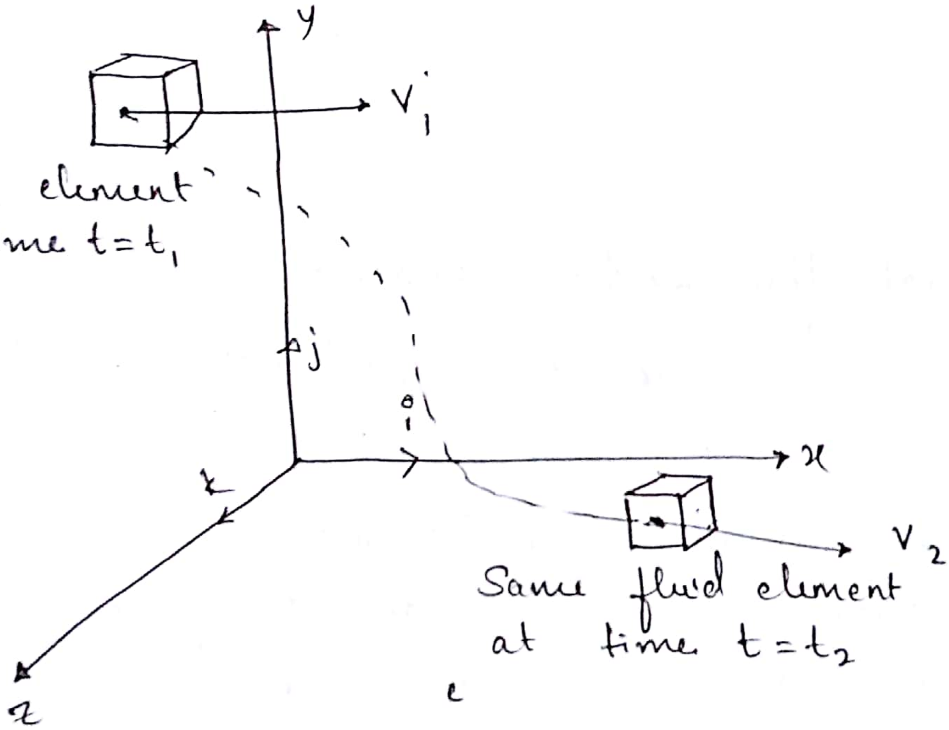
Alternatively it could be moving along ~~with~~ a streamline with velocity vector V equal to flow velocity at each point.

Again instead of looking to whole flow field, ~~we~~ the fundamental principle are applied to infinitesimally small fluid element. The leads to fundamental eqs in Partial differential form. (PD form)

The PD equations obtained from fluid element fixed in ~~nature~~ space are conservative form of eqs

The PD equations obtained from fluid elements moving in flow field ~~is given~~ a are non-conservative form of eqs.

Substantial Derivative [Time Rate of change following a moving fluid element].



→ As seen in fig (e) above, we consider an infinitesimally small fluid element moving with the flow.

→ Here the fluid element is moving through cartesian space, where the unit vectors along x, y & z axes are i, j & k respectively

The vector velocity field in cartesian space is given as

$$\boxed{V = ui + vj + wk} \longrightarrow (1)$$

Where x, y & z components are given as

$$\boxed{\left.\begin{aligned} u &= u(x, y, z, t) \\ v &= v(x, y, z, t) \\ w &= w(x, y, z, t) \end{aligned}\right\} \rightarrow 2}$$

Note we have considered an unsteady ~~field~~ flow where u, v & w are functions of space & time t. In addition to it, ~~if~~ the scalar density field is given by

$$\boxed{\rho = \rho(x, y, z, t)} \rightarrow 3$$

At time $t = t_1$, fluid element is at point 1. At this point & time the density of fluid element is

$$\rho_1 = \rho(x_1, y_1, z_1, t_1) \rightarrow 4$$

At time $t = t_2$, same fluid element has moved to point 2. Hence at time t_2, the density of the fluid element is

$$\rho_2 = \rho(x_2, y_2, z_2, t_2) \longrightarrow 5$$

Since $\rho = \rho(x, y, z, t)$ we can expand this function in taylor series about point 1 as follows

$$\rho_2 = \rho_1 + \left(\frac{\partial \rho}{\partial x}\right)_1 (x_2 - x_1) + \left(\frac{\partial \rho}{\partial y}\right)_1 (y_2 - y_1) + \left(\frac{\partial \rho}{\partial z}\right)_1 (z_2 - z_1)$$

$$+ \left(\frac{\partial \rho}{\partial t}\right)_1 (t_2 - t_1) + \text{higher order terms} \longrightarrow 6$$

$$\boxed{\frac{\rho_2 - \rho_1}{t_2 - t_1} = \left(\frac{\partial \rho}{\partial x}\right)_1 \frac{x_2 - x_1}{t_2 - t_1} + \left(\frac{\partial \rho}{\partial y}\right)_1 \frac{y_2 - y_1}{t_2 - t_1} + \left(\frac{\partial \rho}{\partial z}\right) \frac{z_2 - z_1}{t_2 - t_1} + \left(\frac{\partial \rho}{\partial t}\right)_1} \rightarrow 7$$

The left hand side of eq 7 is physically the average time rate of change of density of fluid element as it moves from point 1 to point 2.

By applying limit as t_2 approaches t_1, this term becomes

$$\boxed{\lim_{t_2 \to t_1} \frac{\rho_2 - \rho_1}{t_2 - t_1} \equiv \frac{D\rho}{Dt}} \rightarrow (8)$$

$\frac{D\rho}{Dt}$ → Instantaneous time rate of change of density of fluid element as it moves through point 1.

$\boxed{\frac{D\rho}{Dt} \rightarrow \text{Substantial derivative.}}$

$\frac{D\rho}{Dt}$ → time rate of change of density of given fluid element as it moves through space.

$\left(\frac{\partial \rho}{\partial t}\right)_1$ → It physically represents time rate of change of density at point 1.

Thus $\frac{D\rho}{Dt}$ differs from $\left(\frac{\partial \rho}{\partial t}\right)$ numerically & physically.

Applying lt $t_2 \to t$ on RHS of eq 7 we have

$$\lim_{t_2 \to t_1} \frac{x_2 - x_1}{t_2 - t_1} = u \; ; \quad \lim_{t_2 \to t_1} \frac{y_2 - y_1}{t_2 - t_1} = v \; ; \quad \lim_{t_2 \to t_1} \frac{z_2 - z_1}{t_2 - t_1} = w$$

$$\boxed{\frac{D\rho}{Dt} = u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z} + \frac{\partial \rho}{\partial t}} \rightarrow (9)$$

From observing eq. 9 closely we have Substantial derivative eq in cartesian co-ordinates

$$\boxed{\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}} \rightarrow (10)$$

The vector operator is given as

$$\boxed{\nabla \equiv i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}} \rightarrow (11)$$

Eq (10) can be represented as below in vector form 2 (5)

$$\boxed{\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + V\cdot\nabla} \rightarrow (12) \qquad V = (ui + vj + wk)$$

[Eq 12 shows substantial derivative in vector form]

Here

$\frac{\partial}{\partial t}$ → local derivative → which physically represents time rate of change of flow field variable at a fixed point.

$\nabla\cdot V$ → convective derivative, which is physically the time rate of change due to movement of fluid element from one location to another in flow field where flow properties are spatially different.

Substantial derivative can be applied to any flow field variable P, T, ρ etc....

Ex:- Local Derivative Convective Derivative

$$\frac{DT}{Dt} \equiv \frac{\partial T}{\partial t} + \nabla\cdot V\,T \equiv \frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z}$$

Here in the above eq the temp of fluid element is changing as element sweeps past a point in the flow because at that point the flow-field temperature itself may be fluctuating with time (the local derivative) & because the ~~flouid~~ fluid element is simply on its way to another point in flow field where temp is differ (convective derivative).

Mathematically what substantial derivative signifies is given below.

We know $\rho = \rho(x, y, z, t)$

Applying chain rule from differential calculus

$$d\rho = \frac{\partial \rho}{\partial x} \cdot dx + \frac{\partial \rho}{\partial y} \cdot dy + \frac{\partial \rho}{\partial z} \cdot dz + \frac{\partial \rho}{\partial t} \cdot dt$$

÷ above eq by dt on both sides

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial \rho}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial \rho}{\partial z} \cdot \frac{dz}{dt} + \frac{\partial \rho}{\partial t}$$

$$u = \frac{dx}{dt} \; ; \quad v = \frac{dy}{dt} \; ; \quad w = \frac{dz}{dt}$$

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z}$$

We see,

$$\frac{d\rho}{dt} = \frac{D\rho}{Dt}$$

, this means substantial derivative is total derivative with respect to time.

Divergence of Velocity

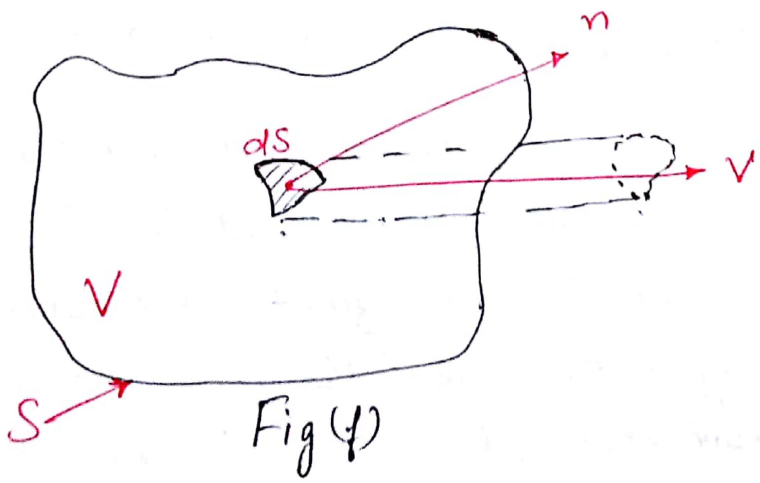


Fig (4)

Consider a control volume moving with the fluid as sketched in figure above. This control volume is always made up of same fluid particles as it moves with the flow; hence its mass is fixed, invariant with time.

However its volume & control surface S are changing with time as it moves to different regions of flow where different values of ρ do exist.

That is moving CV of fixed mass is constantly increasing or decreasing its volume & changing its shape depending on characteristics of the flow.

This control volume is shown in fig (f) is at some instant of time t. Also consider an infinitesimal element of surface dS moving at local velocity V as shown in fig (f).

The change in control volume ΔV, due to just the movement of dS over a time increment Δt, equal to volume of long, thin cylinder with base area dS & altitude $(V\Delta t) \cdot n$ where n is unit vector ⊥ar to surface dS

That is $$\boxed{\Delta V = [(V\Delta t) \cdot n]\, dS = (V\Delta t) \cdot dS} \rightarrow (1)$$

$\boxed{dS \equiv n\, dS}$

Over the time increment Δt, the total change in volume of whole control volume is equal to summation of over total control surface. Apply limit $dS \to 0$, the sum becomes integral

$$\boxed{\iint_S (V\,\Delta t) \cdot dS} \rightarrow (2)$$

If this integral is divided by Δt, the result is physically the time rate of change of control volume denoted by DV/Dt

$$\boxed{\frac{DV}{Dt} = \frac{1}{\Delta t}\iint_S (V \cdot \Delta t) \cdot dS = \iint_S V \cdot dS} \rightarrow (3)$$

Left hand side denotes substantial derivative of volume $\mathcal{V}$, as we are dealing with time rate of change of CV as volume moves with the flow.

Applying divergence theorem from vector Calculus we have on RHS

$$\boxed{\frac{D\mathcal{V}}{Dt} = \iiint_{\mathcal{V}} (\nabla \cdot V)\, d\mathcal{V}} \rightarrow (4)$$

Let us imagine that moving control volume is shrunk to small volume $\delta\mathcal{V}$, essentially becoming an infinitesimal moving fluid element. ~~as~~ sk~~etched~~

Then eq 4 can be written as

$$\boxed{\frac{D(\delta\mathcal{V})}{Dt} = \iiint_{\delta\mathcal{V}} (\nabla \cdot V)\, d\mathcal{V}} \rightarrow (5)$$

Assuming $\delta\mathcal{V}$ is small enough such that $\nabla \cdot V$ is same value throughout $\delta\mathcal{V}$. Then the integral in eq 5 in the limit as $\delta\mathcal{V}$ shrunks to zero is given by $(\nabla \cdot V)\delta\mathcal{V}$

Thus we have

$$\frac{D(\delta\mathcal{V})}{Dt} = (\nabla \cdot V)\delta\mathcal{V} \rightarrow (6)$$

$$V \cdot \nabla = \frac{1}{\delta\mathcal{V}} \frac{D(\delta\mathcal{V})}{Dt} \rightarrow (7)$$

On L.H.S we have divergence of velocity, on RHS its physical meaning in eq. 7

$\nabla \cdot v$ ↳ physically denotes time rate of change of volume of moving fluid element per unit volume.

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