Initial value theorem:

If the Laplace transform of $f(t)$ and $f'(t)$ exists and $L[f(t)] = F(s)$ then

$$\underset{t \to 0}{Lt} [f(t)] = \underset{s \to \infty}{Lt} [s F(s)]$$

Proof:

We know that

$$L[f'(t)] = s L[f(t)] - f(0)$$

$$= s F(s) - f(0)$$

$$\Rightarrow s F(s) = L[f'(t)] + f(0)$$

$$s F(s) = \int_0^\infty e^{-st} f'(t) \, dt + f(0).$$

Taking limit as $s \to \infty$ on both sides we get,

$$\underset{s \to \infty}{Lt} s F(s) = \underset{s \to \infty}{Lt} \left\{ \int_0^\infty e^{-st} f'(t) \, dt + f(0) \right\}$$

$$= \underset{S \to \infty}{Lt} \int_0^\infty e^{-St} f'(t)\,dt + f(0)$$

$$= \int_0^\infty \underset{S \to \infty}{Lt} e^{-St} f'(t)\,dt + f(0)$$

$$= 0 + f(0)$$

$$= \underset{t \to 0}{Lt} f(t)$$

Hence $\underset{t \to 0}{Lt} f(t) = \underset{S \to \infty}{Lt} S F(S)$.

Final value theorem:

If the Laplace transform of $f(t)$ and $f'(t)$ exists and $L[f(t)] = F(S)$ then

$$\underset{t \to \infty}{Lt} [f(t)] = \underset{S \to 0}{Lt} [S F(S)]$$

Proof:

We know that,

$$L[f'(t)] = S L[f(t)] - f(0)$$

$$= S F(S) - f(0)$$

$$\Rightarrow S F(S) = L[f'(t)] + f(0)$$

Taking limit $S \to 0$ on both sides, we get,

$$\underset{S \to 0}{Lt} [S F(S)] = \underset{S \to 0}{Lt} \left\{ \int_0^\infty e^{-St} f'(t)\,dt + f(0) \right\}$$

$$= \int_0^\infty \underset{S \to 0}{Lt} e^{-St} f'(t)\,dt + f(0)$$

$$= \int_0^\infty f'(t)\,dt + f(0)$$

$$= [f(t)]_0^\infty + f(0)$$

$$= f(\infty) - f(0) + f(0) = \underset{t \to \infty}{Lt} f(t)$$

Hence $\underset{t \to \infty}{Lt} f(t) = \underset{S \to 0}{Lt} [S F(S)]$

① Verify the initial and final value theorem for

$f(t) = 1 + e^{-t}(\sin t + \cos t)$

Soln:

$$F(s) = L[1 + e^{-t}\sin t + e^{-t}\cos t]$$

$$= L(1) + L(\sin t)_{s \to s+1} + L(\cos t)_{s \to s+1}$$

$$= \frac{1}{s} + \left(\frac{1}{s^2+1}\right)_{s \to s+1} + \left(\frac{s}{s^2+1}\right)_{s \to s+1}$$

$$= \frac{1}{s} + \frac{1}{(s+1)^2+1} + \frac{s+1}{(s+1)^2+1}$$

$$= \frac{1}{s} + \frac{s+2}{s^2+2s+2}$$

$$\therefore s F(s) = s\left[\frac{1}{s} + \frac{s+2}{s^2+2s+2}\right]$$

Initial value theorem:

$$\underset{t \to 0}{Lt} f(t) = \underset{t \to 0}{Lt}\left[1 + e^{-t}(\sin t + \cos t)\right] = 1 + 1 = 2$$

$$\underset{s \to \infty}{Lt} s F(s) = \underset{s \to \infty}{Lt} s\left[\frac{1}{s} + \frac{s+2}{s^2+2s+2}\right]$$

$$= \underset{s \to \infty}{Lt} 1 + \frac{s^2+2s}{s^2+2s\left(1 + \frac{2}{s^2+2s}\right)}$$

$$= 1 + 1$$

$$= 2$$

Hence $\underset{t \to 0}{Lt} f(t) = \underset{s \to \infty}{Lt} s F(s) = 2$

IVT is verified.

Final value theorem:

$$\underset{t \to \infty}{Lt} f(t) = \underset{t \to \infty}{Lt}\left[1 + e^{-t}(\sin t + \cos t)\right] = 1$$

$$\underset{s\to 0}{Lt}\ sF(s) = \underset{s\to 0}{Lt}\ s\left[\frac{1}{s} + \frac{s+2}{s^2+2s+2}\right]$$

$$= \underset{s\to 0}{Lt}\left[1 + \frac{s^2+2s}{s^2+2s+2}\right] = 1$$

Hence $\underset{t\to\infty}{Lt}\ f(t) = \underset{s\to 0}{Lt}\ sF(s) = 1$, FVT is Verified

Laplace Transform of Some Special functions:

Unit Step function:

The unit step function also called Heavisides unit function is defined as,

$$u(t-a) = \begin{cases} 0, & t < a \\ 1, & t > a \end{cases}$$

This is the unit step functions at $t=a$. It can also be denoted by $H(t-a)$ or $u_a(t)$.

Result:

Laplace Transform of unit step function is $\frac{e^{-as}}{s}$ i.e., $L[u(t-a)] = \frac{e^{-as}}{s}$

Proof:

$$L[u(t-a)] = \int_0^\infty e^{-st}\,u(t-a)\,dt$$

$$= \int_0^a e^{-st}\,u(t-a)\,dt + \int_a^\infty e^{-st}\,u(t-a$$

$$= 0 + \int_a^\infty e^{-st}\,dt$$

$$= \left[\frac{e^{-st}}{-s}\right]_a^\infty$$

$$= \frac{e^{-as}}{s} \quad (s > 0)$$

Second Shifting theorem :

If $L[f(t)] = F(s)$ then

$$L[f(t-a)\,u(t-a)] = e^{-as} F(s)$$

Proof :

$$L[f(t-a)\,u(t-a)] = \int_0^\infty e^{-st} f(t-a)\,u(t-a)\,dt$$

$$= \int_0^a e^{-st} f(t-a)\,u(t-a)\,dt + \int_a^\infty e^{-st} f(t-a)\,u(t-a)\,dt$$

$$= 0 + \int_a^\infty e^{-st} f(t-a)\,u(t-a)\,dt$$

Put $u = t-a \Rightarrow du = dt$

When $t = a$, $u = 0$

$t = \infty$, $u = \infty$

$$\Rightarrow L[f(t-a)\,u(t-a)] = \int_0^\infty e^{-s(a+u)} f(u)\,du$$

$$= e^{-as}\int_0^\infty e^{-us} f(u)\,du$$

$$= e^{-as}\int_0^\infty e^{-st} f(t)\,dt$$

$$= e^{-as} F(s).$$

Unit Impulse function (or) Dirac Delta function:

The dirac delta function is denoted by, $\delta(t-a)$ & it is defined by

$$\delta(t-a) = \lim_{\epsilon \to 0} \delta_\epsilon(t-a)$$