Integral of Laplace Transform (or) Laplace Transform of $\frac{f(t)}{t}$:

If $L[f(t)] = F(s)$ and if $\lim_{t \to 0} \frac{f(t)}{t}$ exists then $L\left[\frac{f(t)}{t}\right] = \int_s^\infty F(s)\, ds$

Proof:

$$L[f(t)] = F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

Integrating w.r.t s from s to ∞, we get,

$$\int_s^\infty F(s)ds = \int_s^\infty \left[\int_0^\infty e^{-st} f(t)dt\right] ds$$

$$= \int_0^\infty \left[\int_s^\infty e^{-st} f(t)\, ds \right] dt$$

$$= \int_0^\infty f(t) \left[\int_s^\infty e^{-st}\, ds \right] dt$$

$$= \int_0^\infty f(t) \left[\frac{e^{-st}}{-s} \right]_s^\infty dt$$

$$= \int_0^\infty f(t) \left[0 - \frac{e^{-st}}{-t} \right] dt$$

$$= \int_0^\infty e^{-st} \frac{f(t)}{t}\, dt$$

$$= L\left[\frac{f(t)}{t} \right]$$

$$\therefore \boxed{L\left[\frac{f(t)}{t} \right] = \int_s^\infty F(s)\, ds.}$$

Problems :

① Find $L\left(\frac{1-\cos t}{t} \right)$

Soln:

$$L\left(\frac{1-\cos t}{t} \right) = \int_s^\infty L[1-\cos t]\, ds$$

$$= \int_s^\infty \{ L(1) - L(\cos t) \}\, ds$$

$$= \int_s^\infty \left[\frac{1}{s} - \frac{s}{s^2+1} \right] ds$$

$$= \left[\log s - \frac{1}{2} \log (s^2+1) \right]_s^\infty$$

$$= \left[\log s - \log (s^2+1)^{1/2}\right]_s^\infty$$

$$= \left[\log \frac{s}{\sqrt{s^2+1}}\right]_s^\infty$$

$$= \left[\log \frac{1}{\sqrt{1+\frac{1}{s^2}}}\right]_s^\infty$$

$$= \log 1 - \log \left[\frac{1}{\sqrt{1+\frac{1}{s^2}}}\right]$$

$$= 0 - \log \frac{s}{\sqrt{s^2+1}}$$

$$= \log \left[\frac{s}{\sqrt{s^2+1}}\right]^{-1}$$

$$= \log \left[\frac{\sqrt{s^2+1}}{s}\right]$$

(2) Find $L\left(\frac{e^{-3t} - e^{-4t}}{t}\right)$.

Soln:

$$L(e^{-3t} - e^{-4t}) = \frac{1}{s+3} - \frac{1}{s+4}$$

$$L\left[\frac{e^{-3t} - e^{-4t}}{t}\right] = \int_s^\infty \left(\frac{1}{s+3} - \frac{1}{s+4}\right) ds$$

$$= \int_s^\infty \left(\frac{1}{s+3} - \frac{1}{s+4}\right) ds$$

$$= \left[\log (s+3) - \log (s+4)\right]_s^\infty$$

$$= \left[\log \left(\frac{s+3}{s+4}\right)\right]_s^\infty = \log \left(\frac{s+4}{s+3}\right)$$

(3) Find $L\left[\frac{1-\cos at}{t}\right]$

Soln:

$$L\left[\frac{1-\cos at}{t}\right] = \int_s^\infty L(1-\cos at)\,ds$$

$$= \int_s^\infty \left[\frac{1}{s} - \frac{s}{s^2+a^2}\right] ds$$

$$= \left[\log s - \frac{1}{2}\log(s^2+a^2)\right]_s^\infty$$

$$= \left[\log \frac{s}{\sqrt{s^2+a^2}}\right]_s^\infty$$

$$= 0 - \log\left(\frac{s}{\sqrt{s^2+a^2}}\right)$$

$$= \log\left(\frac{\sqrt{s^2+a^2}}{s}\right).$$

(4) Find $L\left[\frac{\cos at - \cos bt}{t}\right]$

Soln:

$$L\left[\frac{\cos at - \cos bt}{t}\right] = \int_s^\infty L[\cos at - \cos bt]\,ds$$

$$= \int_s^\infty \left[\frac{a}{s^2+a^2} - \frac{b}{s^2+b^2}\right] ds$$

$$= \frac{1}{2}\left[\log(s^2+a^2) - \log(s^2+b^2)\right]_s^\infty$$

$$= \frac{1}{2}\left[\log \frac{s^2+a^2}{s^2+b^2}\right]_s^\infty$$

$$= \frac{1}{2}\left[0 - \log \frac{s^2+a^2}{s^2+b^2}\right]$$

$$= \frac{1}{2}\log\frac{s^2+b^2}{s^2+a^2}$$

⑤ Find the Laplace transform of $e^{-t}\int_0^t t\cos t\,dt$

Soln:

$$L\left[e^{-t}\int_0^t t\cos t\,dt\right] = \left[L\left(\int_0^t t\cos t\,dt\right)\right]_{s\to s+1}$$

$$\left(\because L\int_0^t f(t)dt = \frac{1}{s}L[f(t)]\right)$$

$$= \left[\frac{1}{s}L(t\cos t)\right]_{s\to s+1}$$

$$= \left[\frac{1}{s}\left(\frac{-d}{ds}L(\cos t)\right)\right]_{s\to s+1}$$

$$= \left[\frac{-1}{s}\frac{d}{ds}\left(\frac{s}{s^2+1}\right)\right]_{s\to s+1}$$

$$= \left[\frac{-1}{s}\left(\frac{s^2+1-2s^2}{(s^2+1)^2}\right)\right]_{s\to s+1}$$

$$= \left[\frac{-1}{s}\left(\frac{1-s^2}{(s^2+1)^2}\right)\right]_{s\to s+1}$$

$$= \left[\frac{s^2-1}{s(s^2+1)^2}\right]_{s\to s+1}$$

$$= \frac{(s+1)^2-1}{(s+1)\left((s+1)^2+1\right)^2}$$

$$= \frac{s^2+2s}{(s+1)(s^2+2s+2)^2}$$

⑥ Evaluate using Laplace transform

$$\int_0^\infty te^{-2t}\sin 3t\,dt$$

Soln:

$$\int_0^\infty t e^{-2t} \sin 3t \, dt = \int_0^\infty e^{-2t} (t \sin 3t) dt$$

$$= \left[\int_0^\infty e^{-st} (t \sin 3t) \, dt \right]_{s=2}$$

$$= \left[L(t \sin 3t) \right]_{s=2}$$

$$= \left[-\frac{d}{ds} L(\sin 3t) \right]_{s=2}$$

$$= \left\{ -\frac{d}{ds} \left(\frac{3}{s^2+9} \right) \right\}_{s=2}$$

$$= \left(\frac{6s}{(s^2+9)^2} \right)_{s=2}$$

$$= \frac{12}{169}.$$