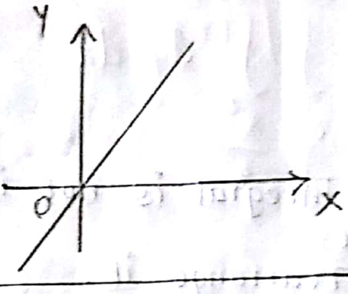
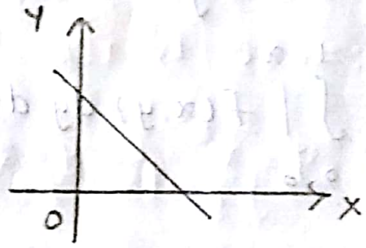
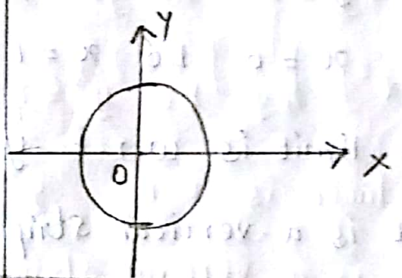
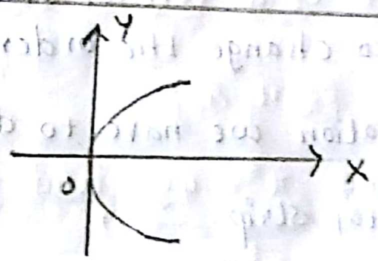
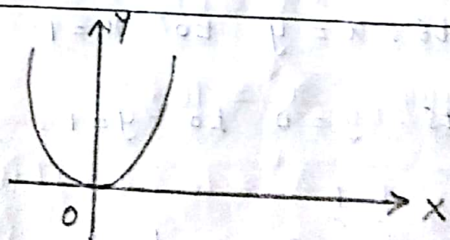
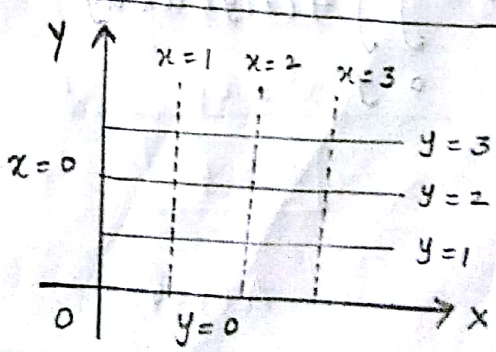
Double integration under a given region

Standard diagrams :

Equation	Graph
1) $y = x$	
2) $x + y = 1$	
3) $x^2 + y^2 = a^2$	
4) $y^2 = 4ax$	
5) $x^2 = 4ay$	
6) $x = a$, $y = b$	

Change of order of integration

(1) Change the order of integration for

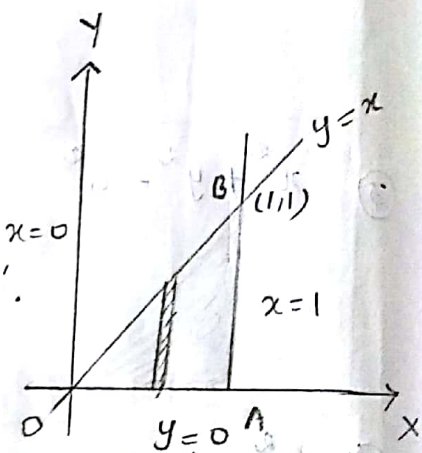
$$\int_0^1 \int_0^x f(x,y)\, dx\, dy$$

Solution :

Given integral is not in the correct order.

Let us rearrange it.

$$I = \int_0^1 \int_0^x f(x,y)\, dy\, dx$$



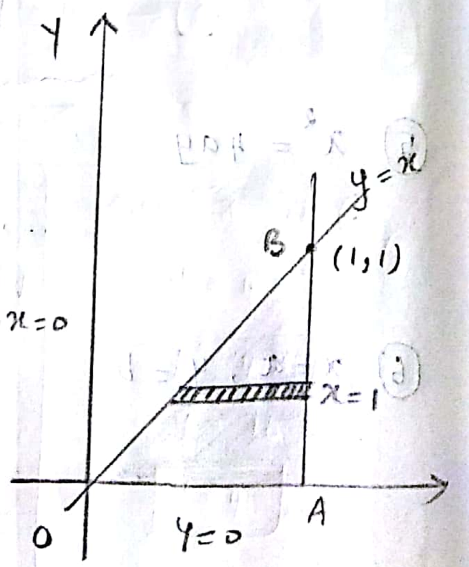
Given : $y = 0$ to $y = x$

$x = 0$ to $x = 1$

Inner limit is w.r.t 'y'.

$\therefore$ It is a vertical strip

Now to change the order of integration we have to draw a horizontal strip.



x limits : $x = y$ to $x = 1$

y limits : $y = 0$ to $y = 1$

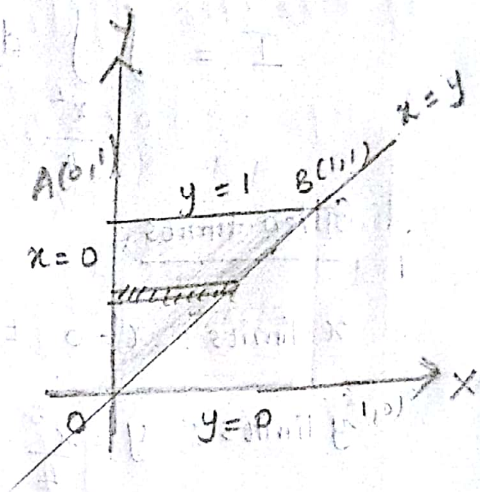
$$\therefore \boxed{I = \int_0^1 \int_y^1 f(x,y)\, dx\, dy}$$

(2) Change the order of integration in

$$\int_0^1 \int_0^y f(x,y)\, dx\, dy$$

Soln:

$$I = \int_0^1 \int_0^y f(x,y)\, dx\, dy$$



Given limits:

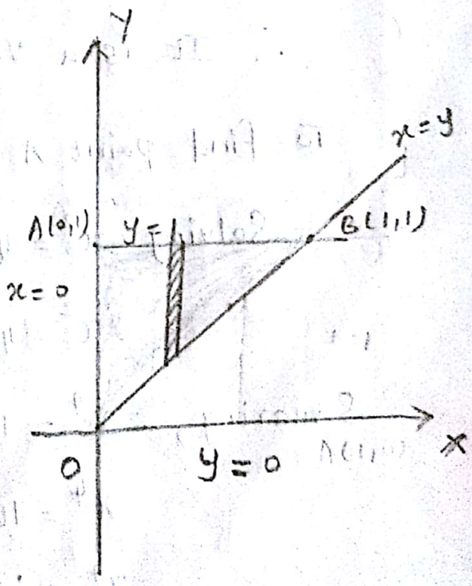
x limit : $x = 0$ to $x = y$

y limit : $y = 0$ to $y = 1$

Inner limit is w.r.t 'x'.

$\therefore$ Given limit is a horizontal strip.

Now to change the order of integration we have to draw a vertical strip.



x limits : $x = 0$ to $x = 1$

y limits : $y = x$ to $y = 1$

$$\therefore \boxed{I = \int_0^1 \int_x^1 f(x,y)\, dy\, dx}$$

③ Evaluate by changing the order of integration in $\int_0^4 \int_{x^2/4}^{2\sqrt{x}} dy\, dx$

Soln:

$$I = \int_0^4 \int_{\frac{x^2}{4}}^{2\sqrt{x}} dy\, dx$$

Given limits:

x limits: $x = 0$ to $x = 4$

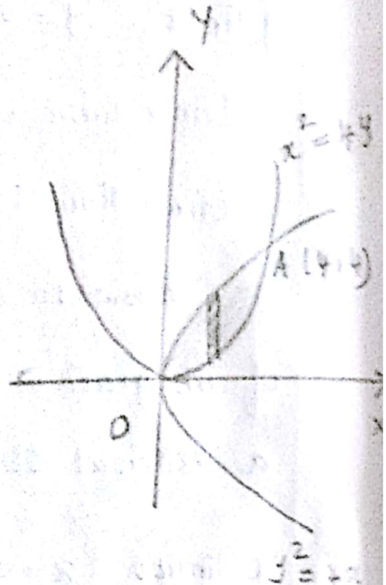
y limits: $y = \frac{x^2}{4}$ to $y = 2\sqrt{x}$

$4y = x^2$ to $y^2 = 4x$

i.e., $x^2 = 4y$ to $y^2 = 4x$

Inner limit is w.r.t 'y'

$\therefore$ It is a vertical strip



To find point A:

Solving $x^2 = 4y \rightarrow (1)$ and $y^2 = 4x \rightarrow (2)$

$x^2 = 4y$

Squaring, $x^4 = 16y^2$

$x^4 = 16(4x)$ (using (2))

$x^4 = 64x$

$x^3 = 64$

$\boxed{x = 4}$

subs x in (2),

$y^2 = 4x$

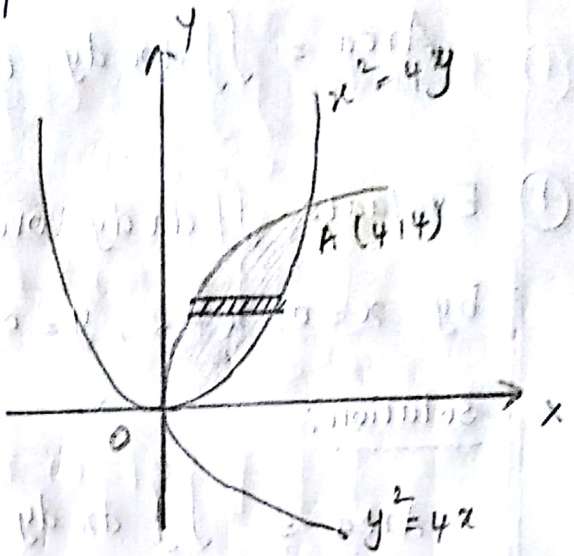
$y^2 = 4 \times 4$

$y^2 = 16$

$\boxed{y = 4}$

A(4,4)

Now to change the order of integration draw a horizontal strip.



x limits:

$x = \frac{y^2}{4}$ to $x = 2\sqrt{y}$

$y = 0$ to $y = 4$

$$\therefore I = \int_0^4 \int_{\frac{y^2}{4}}^{2\sqrt{y}} dx\, dy$$

$$= \int_0^4 [x]_{\frac{y^2}{4}}^{2\sqrt{y}} dy$$

$$= \int_0^4 \left[2\sqrt{y} - \frac{y^2}{4}\right] dy$$

$$= \left[\frac{2y^{3/2}}{3/2} - \frac{1}{4} \cdot \frac{y^3}{3}\right]_0^4$$

$$= \frac{4}{3}(4^{3/2}) - \frac{1}{12}(4^3)$$

$$= \frac{4}{3} \times 8 - \frac{1}{\cancel{12}_3}(\cancel{4} \times 4 \times 4)$$

$4^{3/2} = 4\sqrt{4}$
$= 4 \times 2$
$= 8$

$$= \frac{32}{3} - \frac{16}{3}$$

$$\boxed{I = \frac{16}{3}}$$

Application of double integrals (area) :

$$\text{Area} = \iint dx\,dy \quad (\text{or}) \quad \iint dy\,dx$$

① Evaluate $\iint dx\,dy$ over the region bounded by $x=0,\ x=2,\ y=0,\ y=2$

Solution :

$$\text{Area} = \iint dx\,dy$$

$$= \int_0^2 \int_0^2 dx\,dy$$

$$= \int_0^2 [x]_0^2\,dy$$

$$= \int_0^2 (2-0)\,dy$$

$$= 2\int_0^2 dy$$

$$= 2\,[y]_0^2$$

$$= 2\,(2-0)$$

$$\boxed{\text{Area} = 4 \text{ square units}}$$

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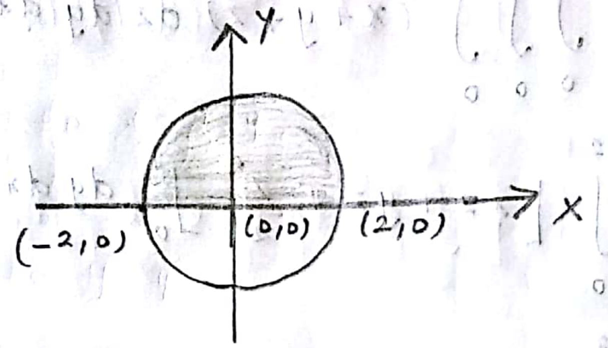
(2) Evaluate $\iint_R dx\,dy$ where R is the shaded region in the figure.



Solution :-

$$\iint_R dx\,dy = \text{Area of the shaded region}$$

$$= \text{Area of semicircle}$$

$$= \frac{1}{2}\pi r^2$$

$$= \frac{1}{\cancel{2}}\pi(2^{\cancel{2}})$$

$$= 2\pi \text{ sq. units}$$

Triple integration :

① Evaluate $\int_0^3 \int_0^2 \int_0^1 (x+y+z)\, dz\, dy\, dx$

$$= \int_0^3 \int_0^2 \left[xz + yz + \frac{z^2}{2} \right]_0^1 dy\, dx$$

$$= \int_0^3 \int_0^2 \left[x(1-0) + y(1-0) + \frac{1}{2}(1-0) \right] dy\, dx$$

$$= \int_0^3 \int_0^2 \left[x + y + \frac{1}{2} \right] dy\, dx$$

$$= \int_0^3 \left[xy + \frac{y^2}{2} + \frac{1}{2} y \right]_0^2 dx$$

$$= \int_0^3 \left[x(2-0) + \frac{1}{2}(2^2 - 0) + \frac{1}{2}(2-0) \right] dx$$

$$= \int_0^3 \left[2x + 2 + 1 \right] dx$$

$$= \int_0^3 (2x+3)\, dx$$

$$= \left[\cancel{2} \cdot \frac{x^2}{\cancel{2}} + 3x \right]_0^3$$

$$= (3^2 - 0 + 3(3-0))$$

$$= 9 + 9$$

$\boxed{I = 18}$

(2) Evaluate $\int_0^1\int_0^2\int_0^3 xyz\,dx\,dy\,dz$

$$= \int_0^1\int_0^2 \left[\int_0^3 x\,dx\right] yz\,dy\,dz$$

$$= \int_0^1\int_0^2 \left[\frac{x^2}{2}\right]_0^3 yz\,dy\,dz$$

$$= \int_0^1\int_0^2 \left[\frac{1}{2}(3^2-0)\right] yz\,dy\,dz$$

$$= \int_0^1\int_0^2 \frac{9}{2}\,yz\,dy\,dz$$

$$= \frac{9}{2}\int_0^1 \left[\int_0^2 y\,dy\right] z\,dz$$

$$= \frac{9}{2}\int_0^1 \left[\frac{y^2}{2}\right]_0^2 z\,dz$$

$$= \frac{9}{2}\int_0^1 \frac{1}{2}(2^2-0)\,z\,dz$$

$$= \frac{9}{\cancel{2}}\int_0^1 \frac{\cancel{4}}{\cancel{2}}\,z\,dz$$

$$= 9\left[\frac{z^2}{2}\right]_0^1$$

$$= \frac{9}{2}[1-0] = \frac{9}{2}$$

$$\boxed{I = \frac{9}{2}}$$

③ Evaluate $\int_0^a \int_0^b \int_0^c e^{x+y+z}\, dx\, dy\, dz$

$$= \int_0^a \int_0^b \int_0^c e^x . e^y . e^z \, dx\, dy\, dz$$

$$= \int_0^a \int_0^b \left[\int_0^c e^x \, dx \right] e^y e^z \, dy\, dz .$$

$$= \int_0^a \int_0^b \left[e^x \right]_0^c e^y e^z \, dy\, dz .$$

$$= \int_0^a \int_0^b (e^c - e^0) e^y e^z \, dy\, dz .$$

$$= (e^c - 1) \int_0^a \left[\int_0^b e^y \, dy \right] e^z \, dz .$$

$$= (e^c - 1) \int_0^a \left[e^y \right]_0^b e^z \, dz$$

$$= (e^c - 1) \int_0^a \left[e^b - e^0 \right] e^z \, dz .$$

$$= (e^c - 1)(e^b - 1) \int_0^a e^z \, dz$$

$$= (e^c - 1)(e^b - 1) \left[e^z \right]_0^a$$

$$= (e^c - 1)(e^b - 1)(e^a - e^0)$$

$$\boxed{I = (e^c - 1)(e^b - 1)(e^a - 1)}$$

Volume of triple integrals :

1) Find the volume of the sphere :

$x^2+y^2+z^2=a^2$ without transformation.

Soln :

Volume = 8 x Volume in the 1^{st} octant.

Z limits :

$$x^2+y^2+z^2=a^2$$

$$z^2=a^2-x^2-y^2$$

$$z=\pm\sqrt{a^2-x^2-y^2}$$

Z varies from 0 to $\sqrt{a^2-x^2-y^2}$

y limits : Put $z=0$ in $x^2+y^2+z^2=a^2$

$$x^2+y^2=a^2$$

$$y^2=a^2-x^2$$

$$y=\pm\sqrt{a^2-x^2}$$

y varies from 0 to $\sqrt{a^2-x^2}$

x limit : Put $y=0$, $z=0$ in $x^2+y^2+z^2=a^2$

$$x^2=a^2$$

$$x=\pm a$$

x varies from 0 to a.

$$V = 8 \iiint dz\, dy\, dx$$

$$= 8 \int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} dz\, dy\, dx$$

$$= 8 \int_0^a \int_0^{\sqrt{a^2-x^2}} [z]_0^{\sqrt{a^2-x^2-y^2}} dy\, dx$$

$$= 8 \int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2-y^2}\, dy\, dx$$

$$= 8 \int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{(\sqrt{a^2-x^2})^2-y^2}\, dy\, dx$$

$$= 8 \int_0^a \left[\frac{y}{2}\sqrt{a^2-x^2-y^2} + \frac{a^2-x^2}{2}\sin^{-1}\left(\frac{y}{\sqrt{a^2-x^2}}\right)\right]_0^{\sqrt{a^2-x^2}} dx$$

Formula :

$$\left[\int \sqrt{a^2-x^2}\, dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right.$$

Here $a = \sqrt{a^2-x^2}$, $x = y$]

$$= 8 \int_0^a \left[0 + \frac{a^2-x^2}{2}\sin^{-1}\left(\frac{\sqrt{a^2-x^2}}{\sqrt{a^2-x^2}}\right) - 0\right] dx$$

$$= 8 \int_0^a \left(\frac{a^2-x^2}{2}\right) \sin^{-1}(1)\, dx$$

$$= 8 \int_0^a \left(\frac{a^2-x^2}{2}\right) \frac{\pi}{2}\, dx$$

$$V = \frac{8\pi}{\cancel{2}\times\cancel{2}} \int_0^a (a^2 - x^2)\,dx$$

$$= 2\pi \left[a^2 x - \frac{x^3}{3} \right]_0^a$$

$$= 2\pi \left[a^2 . a - \frac{a^3}{3} \right]$$

$$= 2\pi \left[a^3 - \frac{a^3}{3} \right]$$

$$= 2\pi \left[\frac{3a^3 - a^3}{3} \right]$$

$$V = 2\pi \left(\frac{2a^3}{3} \right)$$

$$V = \frac{4\pi a^3}{3} \text{ cubic units}$$