Unit - 5

Introduction to Finite Element Method.

The finite element method is that every structure may be considered to be an assemblage of a finite number of individual structural elements.

Most of the techniques are based on the "principle of Discretisation". In this a complex problem of large extent is divided into smaller equivalent units.

Different approaches used for deriving the element properties are,

1. Displacement approach
2. Force approach
3. Hybrid approach
4. Mixed approach.

Finite Elements:

The small elements (1D, 2D (or) 3D) obtained by suitably subdividing the given domain to be analysed.

Nodes (or) Nodal points:

The intersections of the sides of the elements. The nodal points connecting the adjacent elements are called external nodes. The extra nodes used to increase the accuracy of solution but not connected to other elements, are called internal nodes.

Nodal lines and Nodal planes:

The interfaces between the elements.

Linear Elements:

Finite elements with straight sides.

Higher oder Elements:

Finite elements with curved sides.

Continuum:

This is defined as a domain in which matter exists at every point. We may think of a continuum as consisting of an infinite number of connected particles.

Primary unknowns:

For displacement formulation based on potential energy, the main unknowns involved in the formulation are the nodal displacements (δ), which are called the primary unknowns. The corresponding forces would be denoted by (P).

Secondary unknowns:

Those unknowns derived from the primary unknowns are known as secondary unknowns. In displacement based formulations, strains, stresses, moments, shear force are secondary unknowns.

Procedure for Finite Element Analysis:

Step-1: Discretise the continuum:

subdividing the body into a number of elements. In 2D problem, elements would be 2D elements like triangle, rectangle, quadilateral, sector etc.

2. Select Element displacement function:

Strains are functions of displacements, and stresses are functions of strains. In this stress fields are defined with some unknowns.

3. Calulate element Properties:

Calculate the element stiffness matrix interms of unknowns selected inthe previous step.

$$[\delta] = [k][P]$$

4. Obtain the element load vector:

The loading can be concentrated, distributed (or) varying on the element area. This can be in the plane of element (or) normal to the plane of the element.

5. Assemble element properties:

Element stiffness matrices are assembled to generate the structure stiffness matrix.

6. Impose boundary conditions:

Stiffness matrix $[k]$ is $[F] = [k][u]$

$$[u] = [k]^{-1}[F]$$

$[F]$ = force vector $[u]$ = displacements

7. Determine the displacement field, strains, and stresses.

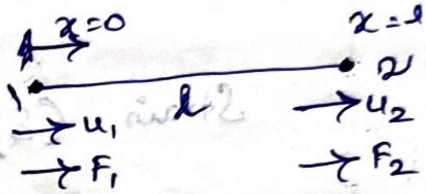
8. Assemblage of elements onto structural continuum.

One-dimensional Finite Element :- Equations :

Let the displacement α at any point P within the element be interpolated from the nodal values of displacement $(\delta)^e$ using the shape function N,

$$\alpha = [N][\delta]^e$$

For a bar element,



$$[\delta]^e = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

where, u_1 - axial displacement
v_1 - transverse displacement

For a beam element,

$$[\delta^e] = \begin{Bmatrix} v_1 \\ dv_1/dx \\ v_2 \\ dv_2/dx \end{Bmatrix}$$

Shape function [N] :

For a bar element,

$$[N] = \left[\left(1 - \frac{x}{l}\right) \quad \left(\frac{x}{l}\right) \right]$$

Shape function derivative, $[B] = [\partial][N] = \frac{dN}{dx}$

$$[K]^e = \int_0^l [B]^T [B] EA \cdot dx$$

Linear bar of finite Element

Consider a linear bar element, It is a one-dimensional line element with two nodes. Each point P within the element is permitted to move only along the axis and the displacement of the entire cross-section is assumed to be same.

Displacement field $\{\alpha\} = u$

Nodal displacement $\{\delta\}^e = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$

Shape function $[N] = [N_1 \quad N_2]$

$$= \left[\left(1 - \frac{x}{l}\right) \left(\frac{x}{l}\right)\right]$$

Strain $\{\varepsilon\} = \varepsilon_x = \frac{du}{dx}$

Shape function derivative $[B] = \begin{bmatrix} \frac{dN_1}{dx} & \frac{dN_2}{dx} \end{bmatrix}$

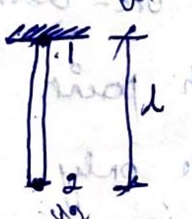
$$= \left(-\frac{1}{l} \quad \frac{1}{l}\right)$$

Element stiffness matrix, $[K]^e = \int_0^l B^T B \, EA \cdot dx$

$$= \int_0^l \begin{pmatrix} -\frac{1}{l} \\ \frac{1}{l} \end{pmatrix} \left(-\frac{1}{l} \quad \frac{1}{l}\right) EA \cdot dx$$

$$= \frac{AE}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

1. A bar subjected to self-weight. Consider a vertically hanging rod of length L, uniform cross-sectional area A, density ρ and Young's modulus E. Find the state of deformation and stress in the rod subjected to a gravity load.



Soln:

1. One-element solution:

$$\frac{AE}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} \qquad K = \frac{F}{u} \quad F = Ku$$

$l = L \quad , \quad u_1 = 0 \quad , \quad F_2 = \rho A g \, L/2$

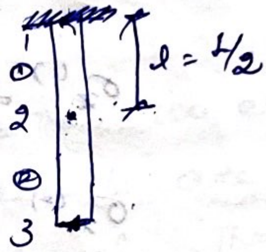
$$u_2 = \left(\frac{\rho A g L}{2} \right) \frac{L}{AE} = \frac{\rho g L^2}{2E}$$

$$u(x) = \left(\frac{\rho g L}{2E} \right) x .$$

Strain, $\epsilon_x = \dfrac{\rho g L}{2E}$

Stress, $\sigma_x = E \epsilon_x = \dfrac{\rho g L}{2}$

2. Two - Element Solution :



Element - ① :

$$\frac{AE}{L/2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} f_1^{(1)} \\ f_2^{(1)} \end{Bmatrix}$$

Element - ② :

$$\frac{AE}{L/2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} f_2^{(2)} \\ f_3^{(2)} \end{Bmatrix}$$

For whole bar element :

$$\frac{AE}{L/2} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1+1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} f_1^{(1)} \\ f_2^{(1)} + f_2^{(2)} \\ f_3^{(2)} \end{Bmatrix}$$

Substituting boundary conditions, $u_1 = 0$

$$f_2^{(1)} = \frac{\rho A g \, L/2}{2} , \quad f_{(2)}^{(2)} = \frac{\rho A g \, L/2}{2} , \quad f_3^{(2)} = \frac{\rho A g \, L/2}{2}$$

$$\frac{AE}{L/2} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} \rho A g \, L/2 \\ \rho A g \, L/4 \end{Bmatrix}$$

$$\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \frac{\rho g L^2}{4E} \begin{Bmatrix} 3/2 \\ 2 \end{Bmatrix}$$

Stress and Strains :

$$\varepsilon_x^{(1)} = [B]\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}^{(1)} = \begin{bmatrix} -1/L/2 & 1/L/2 \end{bmatrix} \begin{Bmatrix} 0 \\ \frac{3\rho g L^2}{8E} \end{Bmatrix}$$

$$= \frac{3}{4}\frac{\rho g L}{E}$$

$$\varepsilon_x^{(2)} = [B]\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix}^{(2)} = \begin{bmatrix} \frac{-1}{L/2} & \frac{1}{L/2} \end{bmatrix} \begin{Bmatrix} 3\rho g L^2/8E \\ \rho g L^2/2E \end{Bmatrix}$$

$$= \frac{1}{4}\frac{\rho g L}{E}$$

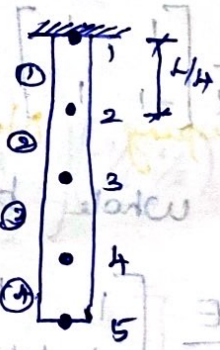
$$\sigma_x^{(1)} = E\varepsilon_x^{(1)} = \frac{3\rho g L}{4}$$

$$\sigma_x^{(2)} = E\varepsilon_x^{(2)} = \frac{\rho g L}{4}$$

Four Element Solution :

Element - 1 :

$$\frac{AE}{L/4}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$



Assembled matrix :

$$\frac{AE}{L/4}\begin{bmatrix} 1+1 & -1 & 0 & 0 \\ -1 & 1+1 & -1 & 0 \\ 0 & -1 & 1+1 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix} = \begin{Bmatrix} 1/2+1/2 \\ 1/2+1/2 \\ 1/2+1/2 \\ 1/2 \end{Bmatrix}\frac{\rho A L g}{4}$$

$$u_2 = \frac{7}{16}\frac{\rho g L^2}{2E}, \quad u_3 = \frac{12}{16}\frac{\rho g L^2}{2E}$$

$$u_4 = \frac{15}{16}\frac{\rho g L^2}{2E}, \quad u_5 = \frac{\rho g L^2}{2E}$$

Strain ϵ :

$$\epsilon^{(1)} = \frac{7}{32}\frac{\rho g L}{E} \qquad \epsilon^{(2)} = \frac{5}{32}\frac{\rho g L}{E}$$

$$\epsilon^{(3)} = \frac{3}{32}\frac{\rho g L}{E} \qquad \epsilon^{(4)} = \frac{1}{32}\frac{\rho g L}{E}$$

Stress, σ :

$$\sigma^{(1)} = \frac{7}{32}\rho g L, \qquad \sigma^{(2)} = \frac{5}{32}\rho g L$$

$$\sigma^{(3)} = \frac{3}{32}\rho g L, \qquad \sigma^{(4)} = \frac{1}{32}\rho g L$$

Element functions :

There are probably more than 100 elements in 2D element. 2-D triangular elements are joining together to form an whole element. These triangular elements having constant strain and these are called as constant strain Elements (CST).

Developing Nodal load vectors :

In FEM, the nature of loading on the continuum have to be converted into concentrated loads at the assigned co-ordinates.

1. Direct method (or) lumped load method
2. Variational method.

2. Solve the matrix equation $[F] = [K][u]$ where, $[F]^T = [100, 120, -10]$ and K is $\begin{bmatrix} 12 & 6 & 2 \\ 6 & 48 & 4 \\ 2 & 4 & 24 \end{bmatrix}$.

Make sure that $u_1 = 0$.

Soln :

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} = [K] \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$\begin{Bmatrix} 100 \\ 120 \\ -10 \end{Bmatrix} = \begin{bmatrix} 12 & 6 & 2 \\ 6 & 48 & 4 \\ 2 & 4 & 24 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$u_1 = 0,$

$$484_2 + 4u_3 = 120$$

$$4u_2 + 24u_3 = -10$$

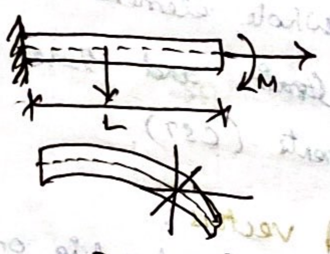
$$u_2 = 2.43, \quad u_3 = -0.845$$

$$[u]^T = [0 \quad 2.43 \quad -0.845]$$

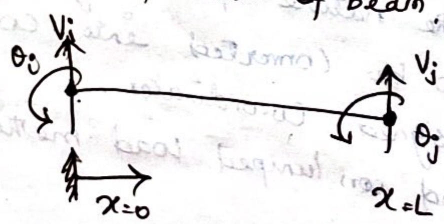
Beam Element:

A truss or rod undergoes only axial deformation and the entire cross section undergoes the same displacement.

A beam element undergoes transverse deflection denoted by v.



Deformation of beam



Shape function:

Nodal values $= V_i, \theta_i, V_j, \theta_j$

Displacement field $V(x) = C_0 + C_1 x + C_2 x^2 + C_3 x^3$

$$\theta = \frac{dv}{dx} = C_1 + 2C_2 x + 3C_3 x^2$$

At $x=0$, $V_i = C_0$

$$\theta_i = C_1$$

At $x=L$, $V_j = C_0 + C_1 L + C_2 L^2 + C_3 L^3$

$$\theta_j = C_1 + 2C_2 L + 3C_3 L^2$$

$$V(x) = N_1 V_i + N_2 \theta_i + N_3 V_j + N_4 \theta_j$$

$$N_1 = 1 - 3x^2/L^2 + 2x^3/L^3$$

$$N_2 = x - 2x^2/L + x^3/L$$

$$N_3 = 3x^2/L^2 - 2x^3/L^3$$

$$N_4 = -x^2/L + x^3/L^2$$

$$V(x) = [N][\delta]^e$$

$$= \begin{bmatrix} N_1 & N_2 & N_3 & N_4 \end{bmatrix} \begin{Bmatrix} V_i \\ \theta_i \\ V_j \\ \theta_j \end{Bmatrix}$$

Element Matrix :

$$\{\varepsilon\} = \varepsilon_x = \frac{du}{dx} = \frac{d}{dx}\left(-z \frac{dv}{dx}\right)$$

$$= -z \frac{d^2 v}{dx^2}$$

$$\varepsilon_x = -z \left[\frac{d^2 N_1}{dx^2} \quad \frac{d^2 N_2}{dx^2} \quad \frac{d^2 N_3}{dx^2} \quad \frac{d^2 N_4}{dx^2} \right] \{\delta\}^e$$

$$= -z \left[\left(\frac{-6}{L^2} + \frac{12x}{L^3}\right) \left(\frac{-4}{L} + \frac{6x}{L^2}\right) \left(\frac{6}{L^2} - \frac{12x}{L^3}\right) \left(\frac{-2}{L} + \frac{6x}{L^2}\right) \right] [\delta]^e$$

Element Stiffness matrix :

$$[K]^e = \int_V [B]^T [B] E \, dv$$

$$= \int_A \int_0^l (-z) \begin{bmatrix} \frac{12x}{L^3} - 6/L^2 \\ 6x/L^2 - 4/L \\ 6/L^2 - 12x/L^3 \\ 6x/L^2 - 2/L \end{bmatrix} E(-z) \left[\left(\frac{12x}{L^3} - \frac{6}{L^2}\right) \left(\frac{6x}{L^2} - \frac{4}{L}\right) \left(\frac{6}{L^2} - \frac{12x}{L^3}\right) \left(\frac{6x}{L^2} - 2/L\right) \right] dA \, dx$$

$$[k]^e = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$

$$\int_A x^2 dA = 1.$$

Gravity Loading :

Gravity loading ($\rho A g$) per unit length

$$\{f\}^e = \int_V [N]^T (\rho g)\, dV$$

$$= \int_0^L [N]^T (\rho g) A \,.\, dx$$

$$= \rho A g \begin{Bmatrix} L/2 \\ L^2/12 \\ L/2 \\ -L^2/12 \end{Bmatrix}$$

This gravity force as a uniformly distributed load. So mentioned as

$$\{f\}^e = \int_0^L [N]^T q_0\, dx$$

$$= \begin{Bmatrix} q_0 L/2 \\ q_0 L^2/12 \\ q_0 L/2 \\ -q_0 L^2/12 \end{Bmatrix}$$

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