Unit Normal Vector

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$$

if $\nabla\phi = x\vec{i} + y\vec{j} + z\vec{k}$

Then $|\nabla\phi| = \sqrt{x^2 + y^2 + z^2}$

Problems:

1) Find the unit normal to the surface $x^2+y^2+z^2=1$ at $(1,1,1)$

Solution:

Given: $\phi = x^2+y^2+z^2-1$

At the point $(1,1,1)$.

unit normal vector $\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$

$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$

$\frac{\partial\phi}{\partial x} = (2x)$

$\frac{\partial\phi}{\partial y} = (2y)$

$\frac{\partial\phi}{\partial z} = (2z)$

$\nabla\phi = \vec{i}(2x)+\vec{j}(2y)+\vec{k}(2z)$

$= 2x\vec{i}+2y\vec{j}+2z\vec{k}$

$(\nabla\phi)_{(1,1,1)} = 2\vec{i}+2\vec{j}+2\vec{k}$

$|\nabla\phi| = \sqrt{2^2+2^2+2^2}$

$= \sqrt{4+4+4}$

$= \sqrt{12}$

$= 2\sqrt{3}$

$|\nabla\phi| = 2\sqrt{3}$.

$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$

$= \frac{2(\vec{i}+\vec{j}+\vec{k})}{2\sqrt{3}}$

$\boxed{\hat{n} = \frac{\vec{i}+\vec{j}+\vec{k}}{\sqrt{3}}}$

2) Find the unit normal to the surface $x^2-y^2+z^2=2$ at $(1,-1,2)$.

Solution:

Given: $\phi = x^2-y^2+z^2-2$ at $(1,-1,2)$

unit normal vector

$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$

$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$.

$\frac{\partial\phi}{\partial x} = 2x$; $\frac{\partial\phi}{\partial y} = 2y$; $\frac{\partial\phi}{\partial z} = 2z$.

$\nabla\phi = 2x\vec{i}+2y\vec{j}+2z\vec{k}$

$(\nabla\phi)_{(1,-1,2)} = 2\vec{i}+2\vec{j}+2\vec{k}$

$|\nabla\phi| = \sqrt{2^2+2^2+4^2}$

$= \sqrt{4+4+16}$

$= \sqrt{24} = \sqrt{6\times4} = 2\sqrt{6}$

$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2\vec{i}+2\vec{j}+4\vec{k}}{2\sqrt{6}}$

$= \frac{2(\vec{i}+\vec{j}+2\vec{k})}{2\sqrt{6}}$

$= \frac{\vec{i}+\vec{j}+2\vec{k}}{\sqrt{6}}$

$\hat{n} = \frac{\vec{i}+\vec{j}+2\vec{k}}{\sqrt{6}}$.

3) Find the unit normal to the surface $x^2+xy+z^2=4$ at $(1,-1,2)$

Soln:

Given: $\phi = x^2+xy+z^2-4$ at $(1,-1,2)$.

unit normal vector

$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$

$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$.

$\frac{\partial\phi}{\partial x} = 2x+y$

$\frac{\partial\phi}{\partial y} = x$

$\frac{\partial\phi}{\partial z} = 2z$.

$\nabla\phi = \vec{i}(2x+y)+\vec{j}(x)+\vec{k}(2z)$

$(\nabla\phi)_{(1,-1,2)} = \vec{i}(2.2+(-1)) + \vec{j}(1)+\vec{k}(2(1))$

$$\nabla\phi = \vec{i}(1) + \vec{j}(1) + \vec{k}(4)$$

$$= \vec{i} + \vec{j} + 4\vec{k}.$$

$$|\nabla\phi| = \sqrt{1^2 + 1^2 + 4^2}$$

$$= \sqrt{18} = \sqrt{6 \times 3} = \sqrt{3 \times 2 \times 3}$$

$$= 3\sqrt{2}$$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{\vec{i} + \vec{j} + 4\vec{k}}{3\sqrt{2}}.$$