Conservative Field:

$\vec{F}$ is said to be conservative if curl $\vec{F}=0$, and $\vec{F}=\nabla\phi$, here ϕ is the scalar potential.

Let $\vec{F}$ be a vector valued function. If there exits a scalar valued function ϕ such that $\vec{F}=\nabla\phi$, then $\vec{F}$ is called conservative and ϕ is called the (scalar) potential of $\vec{F}$.

1) Prove that the vector $\vec{F}=z\vec{i}+x\vec{j}+y\vec{k}$ is solenoidal.

Soln: To prove: $\vec{F}$ is solenoidal,

Then, T.P div $\vec{F}=0$.

$$\text{div }\vec{F}=\nabla\cdot\vec{F}$$

$$=\left(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z}\right)\cdot\left(z\vec{i}+x\vec{j}+y\vec{k}\right)$$

$$=\frac{\partial}{\partial x}(z)+\frac{\partial}{\partial y}(x)+\frac{\partial}{\partial z}(y)$$

$=0+0+0=0$ Hence proved.

HW

Show that the vector $\vec{F}=3y^4z^2\vec{i}+4x^3z^2\vec{j}-3x^2y^2\vec{k}$ is solenoidal.

Irrotational vector:

$$\text{curl }\vec{F}=\nabla\times\vec{F}=0.$$

1) Show that $\vec{F}=yz\vec{i}+zx\vec{j}+xy\vec{k}$ is irrotational.

Soln:

$$\nabla\times\vec{F}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ yz & zx & xy\end{vmatrix}$$

$$=\vec{i}\left(\frac{\partial}{\partial y}(xy)-\frac{\partial}{\partial z}(zx)\right)-\vec{j}\left(\frac{\partial}{\partial x}(xy)-\frac{\partial}{\partial z}(yz)\right)+\vec{k}\left(\frac{\partial}{\partial x}(zx)-\frac{\partial}{\partial y}(yz)\right)$$

$$=\vec{i}(x-x)-\vec{j}(y-y)+\vec{k}(z-z)$$

$$=\vec{i}(0)-\vec{j}(0)+\vec{k}(0)$$

$$\nabla\times\vec{F}=0.$$

$\therefore\ \vec{F}$ is irrotational.

2) Find the value of the constant a, b, c so that the vector

$$\vec{F} = (x+2y+az)\vec{i} + (bx-3y-z)\vec{j} + (4x+cy+2z)\vec{k}$$

is irrotational.

Soln:

Given: $\vec{F}$ is irrotational

Then $\text{curl} \vec{F} = \nabla \times \vec{F} = 0$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+2y+az & bx-3y-z & 4x+cy+2z \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(4x+cy+2z) - \frac{\partial}{\partial z}(bx+3y-z)\right]$$

$$-\vec{j}\left[\frac{\partial}{\partial x}(4x+cy+2z) - \frac{\partial}{\partial z}(x+2y+az)\right]$$

$$+\vec{k}\left[\frac{\partial}{\partial x}(bx-3y-z) - \frac{\partial}{\partial y}(x+2y+az)\right]$$

$$= \vec{i}\left[c-(-1)\right] - \vec{j}\left[4-a\right] + \vec{k}\left[b-(2)\right]$$

$$= \vec{i}(c+1) - \vec{j}(4-a) + \vec{k}(b-2)$$

since $\nabla \times \vec{F} = 0$.

$$\vec{i}(c+1) - \vec{j}(4-a) + \vec{k}(b-2) = 0\vec{i} + 0\vec{j} + 0\vec{k}$$

$c+1=0$	$-(4-a)=0$	$b-2=0$
$c=-1$	$a=4$	$b=2$

$\therefore a=4;\ b=2$ and $c=-1$.

3) Prove that $\vec{F} = (y^2\cos x + z^3)\vec{i} + (2y\sin x - 4)\vec{j} + 3xz^2\vec{k}$ is irrotational and find its Scalar Potential.

Solution:

T.P: $\vec{F}$ is irrotational

$\therefore \text{curl} \vec{F} = \nabla \times \vec{F} = 0$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2\cos x + z^3 & 2y\sin x - 4 & 3xz^2 \end{vmatrix} =$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(3xz^2) - \frac{\partial}{\partial z}(2y\sin x - 4)\right]$$

$$-\vec{j}\left[\frac{\partial}{\partial x}(3xz^2) - \frac{\partial}{\partial z}(y^2\cos x + z^3)\right]$$

$$+\vec{k}\left[\frac{\partial}{\partial x}(2y\sin x - 4) - \frac{\partial}{\partial y}(y^2\cos x + z^3)\right]$$

$= \vec{i}\,[0-0] - \vec{j}\,[3z^2 - 3z^2] + \vec{k}\,[2y\cos x - 2y\cos x]$

$= 0$

$\therefore$ curl $\vec{F} = 0$

$\therefore$ $\vec{F}$ is irrotational

Now to find ϕ such that

$$\vec{F} = \nabla\phi$$

$$(y^2\cos x + z^3)\vec{i} + (2y\sin x - 4)\vec{j} + 3xz^2\vec{k} = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

$$\Rightarrow \frac{\partial\phi}{\partial x} = y^2\cos x + z^3 \quad \text{--- (1)}$$

$$\frac{\partial\phi}{\partial y} = 2y\sin x - 4 \quad \text{--- (2)}$$

$$\frac{\partial\phi}{\partial z} = 3xz^2 \quad \text{--- (3)}$$

Integrating (1), (2) and (3) partially w.r. to x, y, z respectively we get

$$\phi(x,y,z) = y^2\sin x + z^3x + f(y,z)$$

$$\phi(x,y,z) = 2\sin x\,\frac{y^2}{2} - 4y + g(x,z)$$

$$\phi(x,y,z) = 3x\,\frac{z^3}{3} + h(x,y) = xz^3 + h(x,y)$$

$\therefore \phi = y^2\sin x + z^3x + 2\sin x\,\frac{y^2}{2} - 4y + xz^3$

$\phi = y^2\sin x + xz^3 - 4y + c$ is the scalar potential

2) Prove that $\vec{F} = (2x+yz)\vec{i} + (4y+zx)\vec{j} - (6z-xy)\vec{k}$ is solenoidal as well as irrotational. Also find the scalar potential of $\vec{F}$.

Hint:

$$f = \nabla\phi$$

$$\frac{\partial\phi}{\partial x} = 2x + yz$$

$$\frac{\partial\phi}{\partial y} = 4y + zx$$

$$\frac{\partial\phi}{\partial z} = -(6z - xy)$$

$$\phi(x,y,z) = x^2 + xyz + f(y,z)$$

$$= 2y^2 + xyz + g(x,z)$$

$$= -3z^2 + xyz + h(x,y)$$

$$\phi = x^2 + 2y^2 - 3z^2 + xyz + c,$$