Covariance:

→ If X & Y are 2-dimensional R.V, then covariance of X and Y is defined as

$$\text{cov}(X,Y) = E(XY) - E(X)\cdot E(Y)$$

→ If X & Y are independent, then

$$E(XY) = E(X)\cdot E(Y)$$

$$\Rightarrow \text{Cov}(X,Y) = E(X)\cdot E(Y) - E(X)E(Y)$$

$$= 0$$

∴ It is uncorrelated.

Result:

1. $\text{Cov}(aX, bY) = ab\,\text{cov}(X,Y)$
2. $\text{Cov}(aX+b, cY+d) = ac\,\text{Cov}(X,Y)$

Correlation:

$$r_{XY} = \frac{\text{Cov}(X,Y)}{\sigma_X \sigma_Y} \quad (\text{or}) \quad r_{XY} = \frac{\text{cov}(X,Y)}{\sqrt{\text{Var}X}\sqrt{\text{Var}Y}}$$

where σ = standard deviation = $\sqrt{\text{Variance}}$

Regression: Regression is the average relationship between two or more variables.

Regression line: X on Y

$$X - \bar{X} = b_{XY}(Y - \bar{Y})$$

$$b_{XY} = r\frac{\sigma_X}{\sigma_Y}$$

Y on X

$$Y - \bar{Y} = b_{YX}(X - \bar{X})$$

$$b_{YX} = r\frac{\sigma_Y}{\sigma_X}$$

Properties:

→ $\bar{X} = \frac{\Sigma X}{n}$ and $\bar{Y} = \frac{\Sigma Y}{n}$

→ Correlation co-efficient ⇒ $r = \pm\sqrt{b_{XY}\cdot b_{YX}}$

→ Regression co-efficient ⇒ b_{XY} and b_{YX}

1) Let X and Y be discrete R.V with p.m.f $P(x,y) = \frac{x+y}{21}$

and then find Correlation co-efficient $x = 1,2,3;\ y = 1,2$

X \ Y	1	2	$p(x)$
1	$\frac{2}{21}$	$\frac{3}{21}$	$\frac{5}{21}$
2	$\frac{3}{21}$	$\frac{4}{21}$	$\frac{7}{21}$
3	$\frac{4}{21}$	$\frac{5}{21}$	$\frac{9}{21}$
$P(y)$	$\frac{9}{21}$	$\frac{12}{21}$	1

$$\gamma = \frac{Cov(X,Y)}{\sigma_X \sigma_Y} = \frac{E(XY) - E(X)E(Y)}{\sigma_X \sigma_Y}$$

$$E(X) = \Sigma x p(x) = 1\left(\frac{5}{21}\right) + 2\left(\frac{7}{21}\right) + 3\left(\frac{9}{21}\right)$$

$$= \frac{46}{21}$$

$$E(Y) = \Sigma y P(y) = 1\left(\frac{9}{21}\right) + 2\left(\frac{12}{21}\right) = \frac{33}{21}$$

$$E(XY) = \Sigma\Sigma xy P(x,y) = (1)(1)\frac{2}{21} + 1(2)\frac{3}{21} + (1)(3)\frac{4}{21} +$$

$$(2)(1)\frac{3}{21} + (2)(2)\frac{4}{21} + (2)(3)\frac{5}{21}$$

$$= \frac{2+6+12+6+16+30}{21} = \frac{72}{21}$$

$$E(X^2) = \Sigma x^2 p(x) = 1^2\frac{5}{21} + 2^2\frac{7}{21} + 3^2\frac{9}{21} = \frac{5+28+81}{21} = \frac{114}{21}$$

?

$$Var(X) = E(X^2) - [E(X)]^2 = \frac{114}{21} - \left(\frac{46}{21}\right)^2 = \frac{114}{21} - \frac{2116}{441} = \frac{278}{441}$$

$$E(Y^2) = \Sigma y^2 P(y) = 1\left(\frac{9}{21}\right) + 4\left(\frac{12}{21}\right) = \frac{9+48}{21} = \frac{57}{21}$$

$$Var(Y) = E(Y^2) - [E(Y)]^2 = \frac{57}{21} - \left(\frac{33}{21}\right)^2 = \frac{57}{21} - \frac{1089}{441} = \frac{108}{441}$$

$$Cov(X,Y) = E(XY) - E(X)E(Y) = \frac{72}{21} - \frac{46}{21}\,\frac{23}{21} = \frac{72}{21} - \frac{1518}{441}$$

$$= \frac{1512-1518}{441} = \frac{-6}{441}$$

$$r = \frac{Cov(x, y)}{\sigma_x \sigma_y} = \frac{-6/441}{\sqrt{\frac{278}{\cancel{441}}}\ \sqrt{\frac{108}{\cancel{441}}}} = \frac{\cancel{-6}}{\cancel{5327}5}\ \frac{-6}{173.27}$$

$$= -0.035.$$

2. Suppose that the 2-dimensional R.V (X, Y) has the joint p.d.f $f(x,y) = \begin{cases} x+y, & 0<x<1 \text{ \& } 0<y<1 \\ 0, & \text{otherwise} \end{cases}$

i) Obtain the correlation co-efficient b/w X & Y

ii) Check whether X & Y are independent.

M.D.F of X:

$$f(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^1 (x+y)\,dy = \left[xy + \frac{y^2}{2}\right]_0^1 = x + \frac{1}{2}$$

$\therefore f(x) = x + \frac{1}{2}, \quad 0<x<1$

M.D.F of Y:

$$f(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_0^1 (x+y)\,dx = \left[\frac{x^2}{2} + xy\right]_0^1 = \frac{1}{2} + y, \quad 0<y<1$$

$$E(x) = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^1 x\left(x+\frac{1}{2}\right)dx = \int_0^1 x^2 + \frac{x}{2}\,dx$$

$$= \left[\frac{x^3}{3} + \frac{x^2}{4}\right]_0^1 = \frac{1}{3} + \frac{1}{4}$$

$$= \frac{7}{12}$$

$$E(y) = \int_{-\infty}^{\infty} y f(y)\,dy = \int_0^1 y\left(y+\frac{1}{2}\right)dy = \int_0^1 \left(y^2 + \frac{y}{2}\right)dy$$

$$= \left[\frac{y^3}{3} + \frac{y^2}{4}\right]_0^1 = \frac{1}{3} + \frac{1}{4} = \frac{7}{12}$$

$$E(xy) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f(x,y)\,dx\,dy = \int_0^1\int_0^1 xy(x+y)\,dx\,dy$$

$$= \int_0^1\int_0^1 (x^2y + xy^2)\,dx\,dy.$$

$$= \int_0^1 \left[\frac{x^3}{3}y + \frac{x^2y^2}{2}\right]_0^1 dy = \int_0^1 \left[\frac{y}{3} + \frac{y^2}{2}\right] dy$$

$$= \left[\frac{y^2}{6} + \frac{y^3}{6}\right]_0^1$$

$$= \frac{1}{6} + \frac{1}{6} = \frac{2}{6} = \frac{1}{3}$$

$E(x^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx$

$= \int_0^1 x^2\left(x+\frac{1}{2}\right)dx = \int_0^1 \left(x^3 + \frac{x^2}{2}\right)dx$

$= \left[\frac{x^4}{4} + \frac{x^3}{6}\right]_0^1 = \frac{1}{4} + \frac{1}{6} = \frac{10}{24} = \frac{5}{12}$

$E(y^2) = \int_{-\infty}^{\infty} y^2 f(y)\,dy = \int_0^1 y^2\left(y+\frac{1}{2}\right)dy = \int_0^1 \left(y^3 + \frac{y^2}{2}\right)dy$

$= \left[\frac{y^4}{4} + \frac{y^3}{6}\right]_0^1 = \frac{1}{4} + \frac{1}{6} = \frac{10}{24} = \frac{5}{12}$

$Cov(X,Y) = E(XY) - E(X)E(Y)$

$= \frac{1}{3} - \frac{7}{12}\left(\frac{7}{12}\right) = \frac{-1}{144} \neq 0.$

$\therefore$ X & Y are dependent.

$\sigma_X^2 = Var(X) = E(X^2) - [E(X)]^2$

$= \frac{10}{24} - \left(\frac{7}{12}\right)^2 = \frac{10}{24} - \frac{49}{144}$

$\sigma_X^2 = \frac{11}{144}. \Rightarrow \sigma_X = \frac{\sqrt{11}}{12}$

$\sigma_Y^2 = E(Y^2) - [E(Y)]^2 = \frac{5}{12} - \left(\frac{7}{12}\right)^2 = \frac{5}{12} - \frac{49}{144}$

$\sigma_Y^2 = \frac{11}{144} \Rightarrow \sigma_Y = \frac{\sqrt{11}}{12}$

$r = \frac{Cov(X,Y)}{\sigma_X \sigma_Y} = \frac{-1/144}{\frac{\sqrt{11}}{12}\,\frac{\sqrt{11}}{12}} = \frac{-1/144}{\frac{11}{144}} = \frac{-1}{11}$