Continuous Two dimensional Random Variables.

1. Given $f(x,y) = \begin{cases} cx(x-y) & , 0 < x < 2, -x < y < x \\ 0 & , \text{otherwise} \end{cases}$

Find i) c ii) F(x) iii) F(Y/X)

i) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dy\,dx = 1$

$\int_{0}^{2}\int_{-x}^{x} cx(x-y)\,dy\,dx = 1$

$c\int_{0}^{2}\int_{-x}^{x} (x^2 - xy)\,dy\,dx = 1$

$c\int_{0}^{2} \left[x^2 y - x\frac{y^2}{2}\right]_{-x}^{x} dx = 1$

$c\int_{0}^{2} \left[\left(x^3 - \frac{x^3}{2}\right) - \left(-x^3 - \frac{x^3}{2}\right)\right] dx = 1$

$c\int_{0}^{2} \left[x^3 - \cancel{\frac{x^3}{2}} + x^3 + \cancel{\frac{x^3}{2}}\right] dx = 1$

$$c\left[\cancel{2}\frac{x^4}{\cancel{4}_2}\right]_0^2=1$$

$$c\left[\frac{x^4}{2}\right]_0^2=1$$

$$c\,\frac{16}{2}=1$$

$$\Rightarrow 8C=1 \qquad \Rightarrow c=\frac{1}{8}.$$

$$\therefore f(x,y)=\begin{cases}\frac{1}{8}x(x-y), & 0<x<2,\ -x<y<x\\ 0, & \text{otherwise}\end{cases}$$

ii) Marginal density function of X:

$$f(x)=\int_{-\infty}^{\infty}f(x,y)\,dy = \int_{-x}^{x}\frac{1}{8}x(x-y)\,dy$$

$$=\frac{1}{8}\int_{-x}^{x}(x^2-xy)\,dy$$

$$=\frac{1}{8}\left[x^2y-x\frac{y^2}{2}\right]_{-x}^{x}$$

$$=\frac{1}{8}\left[x^3-\cancel{\frac{x^3}{2}}\right]-\left[-x^3-\cancel{\frac{x^3}{2}}\right]$$

$$=\frac{1}{8}\,2x^3 = \frac{x^3}{4},\quad 0<x<2.$$

iii) $$f(Y/X)=\frac{f(x,y)}{f(x)}=\frac{\frac{1}{\cancel{8}_2}\cancel{x}(x-y)}{x^{\cancel{3}}/\cancel{4}}=\frac{x-y}{2x^2}$$

2) The joint probability density function.

$$f(x,y)=\begin{cases}xy^2+\frac{x^2}{8}, & 0\le x\le 2\ \&\ 0\le y\le 1\\ 0, & \text{otherwise}\end{cases}$$

i) $P(X>1\,|\,Y<\frac{1}{2})$ ii) $P(Y<\frac{1}{2}\,|\,X>1)$ iii) $P(X<Y)$ iv) $P(X+Y\le 1)$

$$P(X>1\,/\,Y<1/2)=\frac{P(X>1,\,Y<\frac{1}{2})}{P(Y<1/2)} \quad -①$$

Now, $$P(X>1,\,Y<\tfrac{1}{2})=\int_1^2\int_0^{1/2}\left(xy^2+\frac{x^2}{8}\right)dy\,dx$$

$$=\int_1^2\left[x\frac{y^3}{3}+\frac{x^2}{8}y\right]_0^{1/2}dx \;=\int_1^2\left(\frac{x}{3}\,\frac{1}{8}+\frac{x^2}{8}\,\frac{1}{2}\right)dx$$

$$= \int_1^2 \left(\frac{x}{24} + \frac{x^2}{16}\right) dx$$

$$= \left[\frac{1}{24}\frac{x^2}{2} + \frac{1}{16}\frac{x^3}{3}\right]_1^2$$

$$= \left(\frac{1}{48}4 + \frac{1}{16}\frac{8}{3}\right) - \left(\frac{1}{48} + \frac{1}{48}\right)$$

$$= \frac{4}{48} + \frac{8}{48} - \frac{1}{48} - \frac{1}{48} = \frac{10}{48} = \frac{5}{24} \quad -② $$

$$P(Y < \tfrac{1}{2}) = \int_0^2 \int_0^{1/2} f(x,y)\,dy\,dx = \int_0^2 \int_0^{1/2} \left(xy^2 + \frac{x^2}{8}\right) dy\,dx$$

$$= \int_0^2 \left[\frac{xy^3}{3} + \frac{x^2}{8}y\right]_0^{1/2} dx$$

$$= \int_0^2 \left(\frac{x}{3}\left(\frac{1}{8}\right) + \frac{x^2}{8}\frac{1}{2}\right) dx$$

$$= \int_0^2 \left(\frac{x}{24} + \frac{x^2}{16}\right) dx$$

$$= \left[\frac{x^2}{48} + \frac{x^3}{48}\right]_0^2 = \frac{4}{48} + \frac{8}{48} = \frac{12}{48} = \frac{1}{4} \quad -③$$

sub ② & ③ in ①.

$$\therefore P(X>1 / Y<\tfrac{1}{2}) = \frac{5/24}{1/4} = \frac{5}{24} \times \frac{4}{1} = \frac{5}{6}$$

ii) $$P(Y<\tfrac{1}{2} / X>1) = \frac{P(X>1, Y<\frac{1}{2})}{P(X>1)} \quad -④$$

Now, $$P(X>1) = \int_1^2 \int_0^1 f(x,y)\,dy\,dx = \int_1^2 \int_0^1 \left[xy^2 + \frac{x^2}{8}\right] dy\,dx$$

$$= \int_1^2 \left[\frac{xy^3}{3} + \frac{x^2}{8}y\right]_0^1 dx$$

$$= \int_1^2 \left(\frac{x}{3} + \frac{x^2}{8}\right) dx = \left[\frac{x^2}{6} + \frac{x^3}{24}\right]_1^2$$

$$= \frac{4}{6} + \frac{8}{24} - \frac{1}{6} - \frac{1}{24} = \frac{3}{6} + \frac{7}{24} = \frac{19}{24} \quad -⑤$$

sub ② & ⑤ in ④

$$\therefore P(Y<\tfrac{1}{2} / X>1) = \frac{5/24}{19/24} = \frac{5}{19}$$

iii) $P(X<Y) = \int_0^1 \int_0^y f(x,y)\,dx\,dy = \int_0^1 \int_0^y \left(xy^2 + \frac{x^2}{8}\right) dx\,dy$

$$= \int_0^1 \left[\frac{x^2}{2} y^2 + \frac{x^3}{24}\right]_0^y dy = \int_0^1 \left(\frac{y^4}{2} + \frac{y^3}{24}\right) dy$$

$$= \left[\frac{y^5}{10} + \frac{y^4}{96}\right]_0^1$$

$$= \frac{1}{10} + \frac{1}{96} = \frac{96+10}{960} = \frac{106}{960}$$

iv) $P(X+Y \le 1) = \int_0^1 \int_0^{1-y} f(x,y)\,dx\,dy$

$$= \int_0^1 \int_0^{1-y} \left(xy^2 + \frac{x^2}{8}\right) dx\,dy$$

$$= \int_0^1 \left[\frac{x^2 y^2}{2} + \frac{x^3}{24}\right]_0^{1-y} dy$$

$$= \int_0^1 \left[\frac{(1-y)^2 y^2}{2} + \frac{(1-y)^3}{24}\right] dy$$

$$= \int_0^1 \left[\frac{(1-2y+y^2)y^2}{2} + \frac{1-3y+3y^2-y^3}{24}\right] dy$$

$$= \int_0^1 \frac{12y^2 - 24y^3 + 12y^4 + 1 - 3y + 3y^2 - y^3}{24} dy$$

$$= \int_0^1 \frac{12y^4 - 25y^3 + 15y^2 - 3y + 1}{24} dy$$

$$= \frac{1}{24}\left[\frac{12y^5}{5} - \frac{25y^4}{4} + \frac{15y^3}{3} - \frac{3y^2}{2} + y\right]_0^1$$

$$= \frac{1}{24}\left[\frac{12}{5} - \frac{25}{4} + \frac{15}{3} - \frac{3}{2} + 1\right]$$

$$= \frac{1}{24}\left[\frac{144 - 375 + 300 - 90 + 60}{60}\right]$$

$$= \frac{1}{24}\left[\frac{39}{60}\right] = \frac{39}{1440}$$

$$= 0.027$$

3) The joint p.d.f of the R.V is given by $f(x,y) = kxy\,e^{-(x^2+y^2)}$, $x>0, y>0$. Find i) k ii) Check X & Y are independent

i) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dy\,dx = 1$

Take $x^2 = s$
$2x\,dx = ds$
$dx = \frac{ds}{2x}$
$x\,dx = ds/2$
$dx = \frac{ds}{2\sqrt{s}}$

$y^2 = t$
$2y\,dy = dt$
$dy = \frac{dt}{2y}$
$y\,dy = dt/2$
$dy = \frac{dt}{2\sqrt{t}}$

$\int_0^{\infty}\int_0^{\infty} kxy\,e^{-(x^2+y^2)}\,dy\,dx = 1$

$k\int_0^{\infty}\int_0^{\infty} xy\,e^{-x^2}e^{-y^2}\,dy\,dx = 1$

$k\int_0^{\infty}\int_0^{\infty} e^{-s}e^{-t}\frac{ds}{2}\frac{dt}{2} = 1$

$\frac{k}{4}\int_0^{\infty} e^{-s}\left[\frac{e^{-t}}{-1}\right]_0^{\infty} ds = 1$

$-\frac{k}{4}\int_0^{\infty} e^{-s}\left[e^{-\infty} - e^{0}\right] ds = 1$

$-\frac{k}{4}\int_0^{\infty} e^{-s}(0-1)\,ds = 1$

$\frac{k}{4}\int_0^{\infty} e^{-s}\,ds = 1$

$\frac{k}{4}\left[\frac{e^{-s}}{-1}\right]_0^{\infty} = 1 \Rightarrow -\frac{k}{4}\left[e^{-\infty} - e^{0}\right] = 1 \Rightarrow -\frac{k}{4}(0-1) = 1$

$\Rightarrow \frac{k}{4} = 1 \quad \Rightarrow \boxed{k=4}$

ii) To prove: $f(x,y) = f(x)\cdot f(y)$

$f(x) = \int_0^{\infty} f(x,y)\,dy = \int_0^{\infty} 4xy\,e^{-(x^2+y^2)}\,dy = 4xe^{-x^2}\int_0^{\infty} ye^{-y^2}\,dy$

$= 4xe^{-x^2}\int_0^{\infty} e^{-t}\frac{dt}{2} = \frac{4xe^{-x^2}}{2}\left[\frac{e^{-t}}{-1}\right]_0^{\infty}$

$= -2xe^{-x^2}(e^{-\infty} - e^{0}) = -2xe^{-x^2}(0-1)$

$f(x) = 2xe^{-x^2}, \quad x>0$

IIIly $f(y) = \int_0^{\infty} f(x,y)\,dx = \int_0^{\infty} 4xy\,e^{-(x^2+y^2)}\,dx = 4ye^{-y^2}\int_0^{\infty} xe^{-x^2}\,dx$

$= 4ye^{-y^2}\int_0^{\infty} e^{-s}\frac{ds}{2} = \frac{4ye^{-y^2}}{2}\left[\frac{e^{-s}}{-1}\right]_0^{\infty} = -2ye^{-y^2}\left[e^{-\infty} - e^{0}\right]$

$= -2ye^{-y^2}[0-1] = 2ye^{-y^2}, \quad y>0$

$\therefore f(x) \cdot f(y) = 2x e^{-x^2} \cdot 2y e^{-y^2}$

$= 4xy e^{-(x^2+y^2)}, \quad x>0, y>0$

$= f(x,y).$

$\therefore x$ & y are independent.

4) If the joint density function of x & y is given by

$$F(x,y) = \begin{cases} (1-e^{-x})(1-e^{-y}), & x>0, y>0 \\ 0, & \text{otherwise} \end{cases}$$

Prove that x & y are independent.

W.K.T, $f(x,y) = \frac{\partial^2}{\partial x \partial y} F(x,y) = \frac{\partial^2}{\partial x \partial y}\left[1 - e^{-y} - e^{-x} + e^{-(x+y)}\right]$

$= \frac{\partial}{\partial x}\left[-e^{-y}(-1) + e^{-x} e^{-y}(-1)\right]$

$= \frac{\partial}{\partial x}\left[e^{-y} - e^{-x} e^{-y}\right] = -e^{-y} e^{-x}(-1)$

$f(x,y) = e^{-(x+y)}$

To prove: $f(x,y) = f(x) \cdot f(y)$.

Now, $f(x) = \int_0^\infty f(x,y)\, dy = \int_0^\infty e^{-(x+y)} dy = \int_0^\infty e^{-x} e^{-y} dy$

$= e^{-x}\left[\frac{e^{-y}}{-1}\right]_0^\infty = -e^{-x}\left[e^{-\infty} - e^0\right] = -e^{-x}(0-1)$

$= e^{-x}$

$f(y) = \int_0^\infty f(x,y)\, dx = \int_0^\infty e^{-(x+y)} dx = \int_0^\infty e^{-x} e^{-y} dx$

$= e^{-y} \int_0^\infty e^{-x} dx = e^{-y}\left[\frac{e^{-x}}{-1}\right]_0^\infty = -e^{-y}\left[e^{-\infty} - e^0\right]$

$= -e^{-y}[0-1] = e^{-y}$

$\therefore f(x) \cdot f(y) = e^{-x} \cdot e^{-y}$

$= e^{-(x+y)}$

$= f(x,y)$

$\therefore x$ & y are independent.