Subject Code & Name: 19AST203 Aircraft Structural Mechanics

TOPIC: Castigliano's theorems

2.2 CASTIGLIANO'S THEOREMS

2.2.1 Theorem-I

In any structure subjected to any load system, the deflection at any point is given by the partial differential coefficient of the total strain energy stored with respect to force acting at a point.

i.e., $x_i = \dfrac{\partial U}{\partial P_i}$ and $\theta_i = \dfrac{\partial U}{\partial M_i}$

The proof for this theorem for a point load is given below.

Let x_i be the displacement or deflection at the respective applied load.

The total strain energy, $U = \frac{1}{2} P_i x_i$

$$U = \frac{1}{2}[P_1 x_1 + P_2 x_2 + \cdots + P_n x_n] \quad \text{... (2.11)}$$

Now, P_1 alone be increased to δP_1 gradually then $\delta x_1, \delta x_2, \delta x_3$ ... δx_n be the further deflections.

∴ The resultant work to be added to P_1 alone, and it becomes,

$$\delta U = \left(P_1 + \frac{\delta P_1}{2}\right)\delta x_1 + P_2\,\delta x_2 + P_3\,\delta x_3 + \cdots + P_n\,\delta x_n \quad \text{... (2.12)}$$

If the product of the two small quantities is neglected, the above equation is reduced to,

$$\delta U = P_1\,\delta x_1 + P_2\,\delta x_2 + \ldots + P_n\,\delta x_n \quad \text{... (2.13)}$$

Further, if the loads $(P_1 + \delta P_1)$, P_2, P_3, ..., P_n were all applied gradually from zero, the total strain energy would have been $(U + \delta U)$ and the corresponding deflections would be,

$$(x_1 + \delta x_1), (x_2 + \delta x_2), (x_3 + \delta x_3), \ldots, (x_n + \delta x_n)$$

∴ The total strain energy is given by,

$$(U + \delta U) = \frac{1}{2}(P_1 + \delta P_1)(x_1 + \delta x_1) + \frac{1}{2}P_2(x_2 + \delta x_2) + \cdots \frac{1}{2}P_n(x_n + \delta x_n)$$

Subtracting equation 2.11 from equation 2.14, we get,

$$\delta U = \frac{1}{2}x_1\,\delta P_1 + \frac{1}{2}P_1\,\delta x_1 + \frac{1}{2}P_2\,\delta x_2 + \cdots + \frac{1}{2}P_n\,\delta x_n$$

$$2\delta U = x_1\,\delta P_1 + P_1\,\delta x_1 + P_2\,\delta x_2 + P_3\,\delta x_3 + \ldots + P_n\,\delta x_n \quad \ldots (2.15)$$

Subtracting equation 2.13 from equation 2.15,

$$\delta U = x_1\,\delta P_1$$

$$\frac{\delta U}{\delta P_1} = x_1$$

When, $\delta P_1 \to 0$ (in the limiting case),

$$\boxed{\frac{\partial U}{\partial P_1} = x_1} \quad \ldots (2.16)$$

Similarly, $\frac{\partial U}{\partial P_2} = x_2;\ \frac{\partial U}{\partial P_3} = x_3;\ \frac{\partial U}{\partial P_n} = x_n$

If we proceed in the same way for couples, we shall get,

$$\boxed{\theta_1 = \frac{\partial U}{\partial M_1}} \quad \ldots (2.17)$$

and $\theta_2 = \frac{\partial U}{\partial M_2},\ \theta_3 = \frac{\partial U}{\partial M_3},\ \ldots,\ \theta_n = \frac{\partial U}{\partial M_n}$.

From equation 2.16 and 2.17, the Castigliano's 1st theorem is proved.

2.2.2 Theorem-II

In any structure subjected to any load system, the load acting at point is given by the partial differential coefficient of the total strain energy stored with respect to the deflection at a point.

i.e., $$P_i = \frac{\partial U}{\partial x_i} \text{ and } M_i = \frac{\partial U}{\partial \theta_i}$$

The proof for this theorem for a concentrated load is given below.

Let P_i be the load applied on a corresponding deflection.

$\therefore$ The total strain energy, $U = \frac{1}{2} P_i x_i$

$$U = \frac{1}{2} P_1 x_1 + \frac{1}{2} P_2 x_2 + \cdots + \frac{1}{2} P_n x_n \quad \text{... (2.11)}$$

Note: The above equation is same equation already we obtained in theorem-I.

Now, imagine that x_1 alone to be increased to δx_1 gradually and then $\delta P_1, \delta P_2, \ldots \delta P_n$ be the further loads.

$$\therefore \delta U = \left(x_1 + \frac{\delta x_1}{2} \right) \delta P_1 + \delta P_2 \cdot x_2 + \delta P_3 \cdot x_3 + \cdots + \delta P_n \cdot x_n \quad \text{... (2.18)}$$

Further, if the deflections $(x_1 + \delta x_1), x_2, \ldots, x_n$ were applied all deflections from zero, the total strain energy would have been $(U + \delta U)$ and the corresponding loads would be,

$$(P_1 + \delta P_1), (P_1 + \delta P_2), \ldots, (P_n + \delta P_n)$$

$\therefore$ The total strain energy is given by,

$$(U + \delta U) = \frac{1}{2}(P_1 + \delta P_1)(x + \delta x_1) + \frac{1}{2} P_2 (x_2 + \delta x_2) + \frac{1}{2} P_n (x_n + \delta x_n) \quad \text{...(2.19)}$$

Subtracting equation 2.11 from equation 2.19, we get,

$$\delta U = \frac{1}{2}P_1\,\delta x_1 + \frac{1}{2}x_1\,\delta P_1 + \frac{1}{2}x_2\,\delta P_2 + \cdots + \frac{1}{2}x_n\,\delta P_n$$

$$2\delta U = P_1\,\delta x_1 + x_1\,\delta P_1 + x_2\,\delta P_2 + \ldots + x_n\,\delta P_n \quad \ldots (2.20)$$

Subtracting equation 2.18 from equation 2.20, we get

$$\delta U = P_1\,\delta x_1$$

$$\frac{\delta U}{\delta x_1} = P_1$$

When, $\delta x_1 \rightarrow 0$ (in the limiting case),

$$\boxed{\frac{\partial U}{\partial x_1} = P_1} \quad \ldots (2.21)$$

Similarly, $\frac{\partial U}{\partial x_2} = P_2$; $\frac{\partial U}{\partial x_3} = P_3$; $\frac{\partial U}{\partial x_n} = P_n$

If we proceed in the same way for angular deflections, we shall get,

$$\boxed{M_1 = \frac{\partial U}{\partial \theta_1}} \quad \ldots (2.22)$$

and $M_2 = \frac{\partial U}{\partial \theta_2}$, $M_3 = \frac{\partial U}{\partial \theta_3}$, ..., $M_n = \frac{\partial U}{\partial \theta_n}$

From equations 2.21 and 2.22, the Castigliano's 2nd theorem is proved.

2.2.3 Applications of Castigliano's theorems

Castigliano's theorems can be used in the following cases.

- To determine the displacements of complicated structures.
- To determine the deflections of curved beams, springs, etc.
- To find the deflection of beams due to shear or bending, if the total strain energy due to shearing forces or bending moments is known.

Note:
To remember the following points while applying Castigliano's theorems

- Treat all the loads and couples/moments as variables and carryout the partial differentiation.
- Substitute the known numerical values of different loads and couples in the above equation.
- To find out the deflection or rotation at a point of the structure where there is no load or couple acting, then it may be assumed that a dummy load 'P' or dummy couple/moment is acting at that point and give a value zero at the end.

2.3 MAXWELL'S RECIPROCAL THEOREM: (ENERGY THEOREM)

It states that, the workdone by the first system of loads due to displacements caused by a second system of loads equals the workdone by the second system of loads due to displacements caused by the first system of loads.

Let the point loads $P_1, P_2, P_3, P_4 \ldots P_n$ acting on a elastic body constrained in a space. Then the strain energy due to this force system is given by,

$$U_A = \sum_{i=1}^{n} \frac{1}{2} P_i \cdot \delta_i \quad \text{and} \quad U_B = \sum_{j=1}^{m} \frac{1}{2} P_j \cdot \delta_j \quad (2.23)$$

Where δ_i are the corresponding deflections under the loads P_i and δ_j are the corresponding deflections under the loads P_j.