Subject Code & Name**: 19AST203 Aircraft Structural Mechanics**

TOPIC: **General Method Bending equation**

Bending Of Beams with Non-Symmetrical Cross Sections

The majority of aircraft structural components consist of beams with non-symmetrical cross section acting in bending. For this reason an expression needs to be derived to allow for the determination of the stresses induced by bending moments to such sections.

SIGN CONVENTION AND NOTATION

Look at the Oxyz system of axis, with an arbitrary beam parallel to the z-axis:

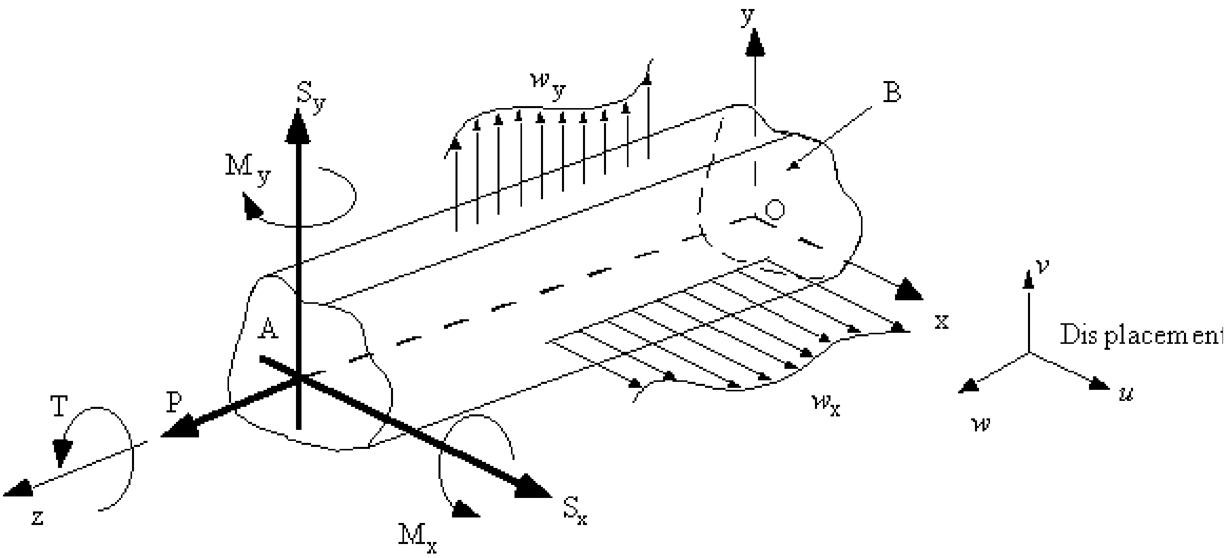

Figure Notation and sign convention for positive forces, moments and displacements.

Where :

T = Torque

M = Bending Moment

S = Shear Force

w = Distributed load

P = Axial or direct load

u,v,w = Axial displacements

All of these externally applied loads are positive in the direction indicated in the figure. Internal moment and forces applied to face A are in the same direction and sense as externally applied loads. However on face B, the positive internal moments and forces are in the opposite sense.

RESOLUTION OF BENDING MOMENTS

A bending moment M applied in any plane parallel to the z-axis can be resolved into M_x and M_y components by normal vector rules.

By doing it in a visual way it will be easier to see:

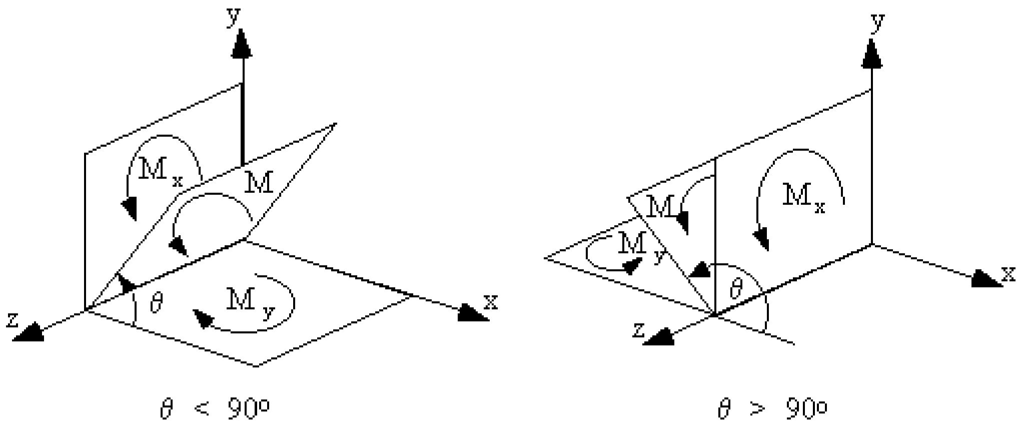

Figure 9: Resolved bending moment about x and y axis.

From Figure 9, the following relationships can be obtained:

$$M_x = M \sin \theta$$

$$M_y = M \cos \theta$$

and that these moments can have different sign depending on the value of θ. For example if $\theta > \pi/2$, M_x is positive and M_y is negative.

STRESS DISTRIBUTION DUE TO BENDING (ETB-NonSym)

Consider a beam of arbitrary non-symmetrical cross section, which supports bending moments Mx and My, bending about some axis in the cross section. This is the plane of no bending stress, called Neutral Axis (N.A.).

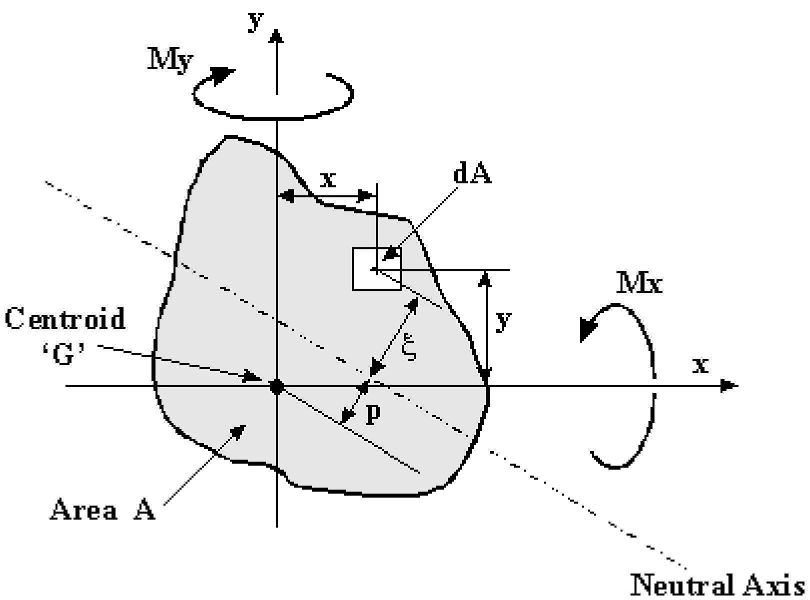

Figure 10: Determination of Neutral Axis location.

Let the axis origin coincide with the centroid G of the cross section, and that the neutral axis is a distance *p* from G.

The direct stress σ_z on element $\mathbf{d_A}$ at point (x,y) and distance ξ from the neutral axis is:

$$\sigma_z = E\varepsilon_z \quad (3.1)$$

Look at the beam in a plane parallel to the neutral axis with two segments ***ij*** and ***kl*** which are of equal length when the beam is undeflected:

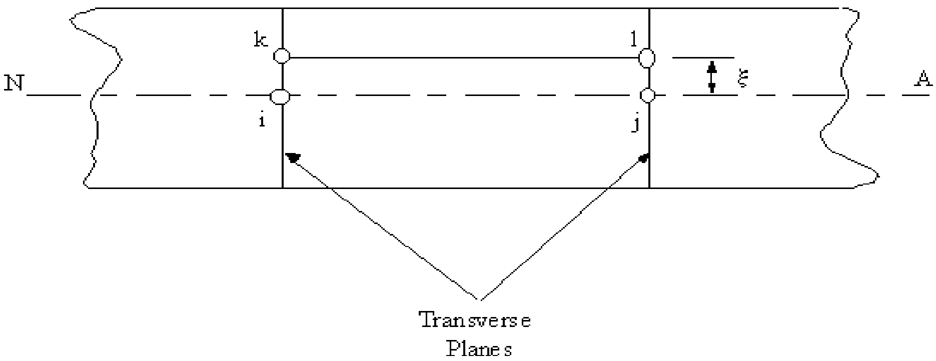

Figure 11: Side view of undeflected beam with segments ***ij*** and ***kl*** marked.

Once the beam has been deflected this section will look like this:

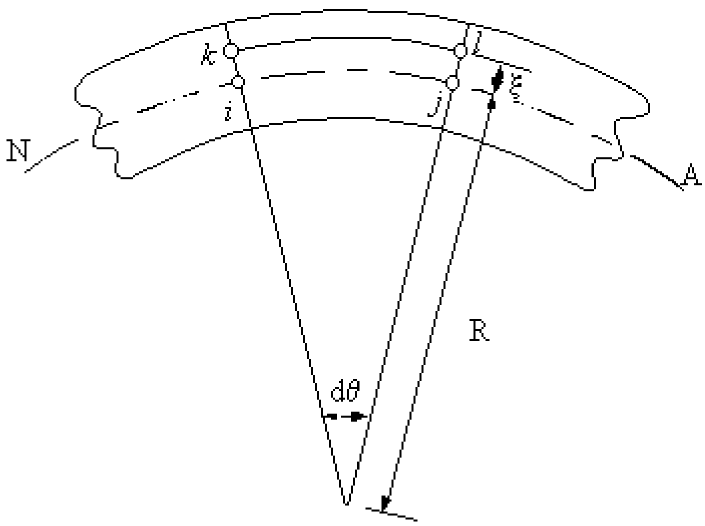

Figure : Deflected beam.

where:

R = the radius of curvature

dθ = angle between planes ***ik*** and ***jl***

The strain in plane ***kl*** can be defined as:

$$\varepsilon = \frac{\Delta L}{L} = \frac{kl - ij}{ij}$$

with

$$ij = Rd\theta$$

and

$$kl = (R + \xi)d\theta$$

giving

$$\varepsilon = \frac{\xi}{R}$$

By substituting back into the stress equation it gives:

$$\sigma_z = \frac{\xi E}{R}$$

Now that a stress equation has been obtained, it is necessary to satisfy both rotational and linear equilibrium at the ends of the beam. That is, the summation of the forces through the depth of the beam is equal to '0', and the summation of the moments through the depth of the beam is equal the applied moments Mx and My.

As the beam supports pure bending, the resultant load on the end section must be zero. Hence

$$\sum F_z = \int_A \sigma_z dA = 0$$

Substituting equation 3.2 gives:

$$\int_A \frac{E\xi}{R} dA = \frac{E}{R}\int_A \xi dA = \int_A \xi dA = 0$$

This equation defines the location of the centroid of the section, it follows that the neutral axis must pass through the centroid.

Rather than use this equation to find the location of the centroid, it is much easier to locate the centroid about the xy-axis by using equation 3.3.

$$\bar{x} = \frac{\sum \bar{x}_i * A_i}{\sum A_i} \qquad \bar{y} = \frac{\sum \bar{y}_i * A_i}{\sum A_i} \qquad \textbf{(3.3)}$$

In order to have moment equilibrium, it is necessary to re-draw Figure 10 but with the axis passing through the centroid.

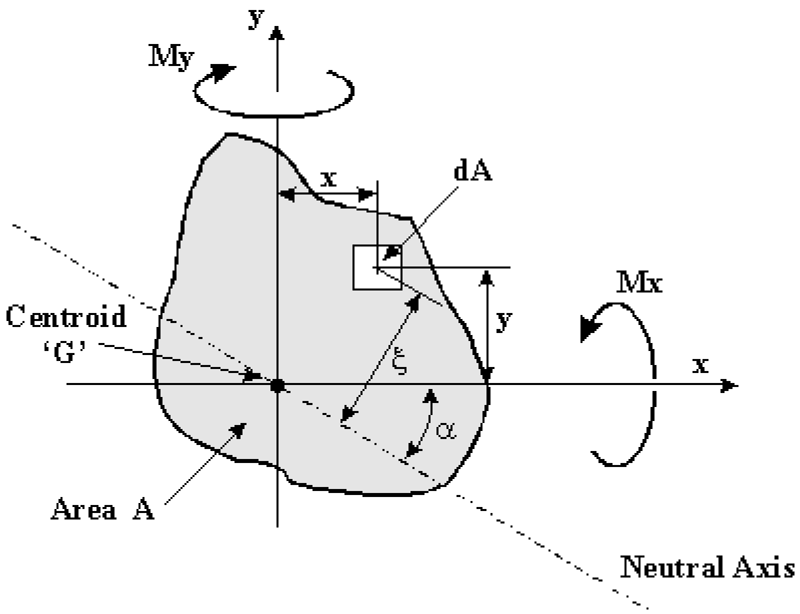

Figure: Beam section with Neutral Axis passing through centroid.

In order to see this in more detail, Figure 14 shown a close up of the axis and the area **dA**.

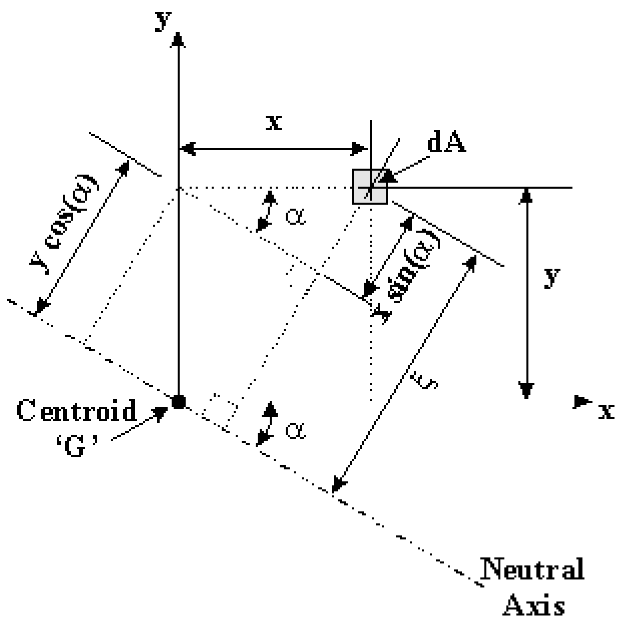

Figure: Detail of area **dA** in beam cross-section.

If the inclination of the Neutral Axis (N.A.) is at an angle from the x-axis then :

$$\xi = x \sin \alpha + y \cos \alpha$$

and substituting into 3.2 gives:

$$\sigma_z = \frac{E}{R}\left(x \sin(\alpha) + y \cos(\alpha)\right)$$

The moment resultants have the same sense as the applied moments, hence:

$$M_x = \int_A \sigma_z y dA \quad , \quad M_y = \int_A \sigma_z x dA$$

and substituting equation 3.4 into equations 3.5 gives:

$$M_x = \frac{E}{R}\int_A \left[xy \sin(\alpha) + y^2 \cos(\alpha)\right] dA$$

and

$$M_y = \frac{E}{R}\int_A \left[x^2 \sin(\alpha) + x \cos(\alpha)\right] dA$$

Both the sin α and cos α terms are not a function of dA, so they can be removed from the integration. What remain are terms which only have to do with the characteristics of the cross sectional shapes of the beam, and these are just the 2^{nd} moments of area of the beam.

The second moments of area about the xy axes are:

$$I_{xx} = \int_A y^2 dA \quad , \quad I_{yy} = \int_A x^2 dA \quad , \quad I_{xy} = \int_A xy\, dA \qquad \textbf{(3.7)}$$

which gives:

$$M_x = \frac{E\sin(\alpha)}{R} I_{xy} + \frac{E\cos(\alpha)}{R} I_{xx}$$
$$M_y = \frac{E\sin(\alpha)}{R} I_{yy} + \frac{E\cos(\alpha)}{R} I_{xy}$$

solving simultaneously gives,

$$\frac{E\cos(\alpha)}{R} = \frac{M_x I_{yy} - M_y I_{xy}}{I_{xx} I_{yy} - I_{xy}^2}$$

Substituting into Equation **(3.5)** gives:

$$\sigma_z = \left(\frac{M_y I_{xx} - M_x I_{xy}}{I_{xx} I_{yy} - I_{xy}^2}\right) x + \left(\frac{M_x I_{yy} - M_y I_{xy}}{I_{xx} I_{yy} - I_{xy}^2}\right) y \qquad \textbf{(3.8)}$$

By defining the terms ***Effective Bending Moment***, as:

$$\bar{M}_x = \frac{M_x - M_y I_{xy} / I_{yy}}{1 - I_{xy}^2 / I_{xx} I_{yy}} \qquad \textbf{(3.9)}$$

and

$$\bar{M}_y = \frac{M_y - M_x I_{xy} / I_{xx}}{1 - I_{xy}^2 / I_{xx} I_{yy}} \qquad \textbf{(3.10)}$$

Equation 3.8 can be re-written as follows:

$$\sigma_z = \frac{\bar{M}_x}{I_{xx}} y + \frac{\bar{M}_y}{I_{yy}} x \qquad \textbf{(3.11)}$$