Milne's Predictor and corrector Methods:

Milne's method is a multistep method that first predicts a value for y_{n+1} from 3 past values of the derivatives. The past values are computed using either RK method (or) Taylor series method.

Milne's predictor and Corrector formula is,

$$y_{n+1,P} = y_{n-3} + \frac{4h}{3}\left(2y'_{n-2} - y'_{n-1} + 2y'_n\right)$$

$$y_{n+1,C} = y_{n-1} + \frac{h}{3}\left(y'_{n-1} + 4y'_n + y'_{n+1}\right)$$

Q. Solve $y' = x - y^2$, $0 \le x \le 1$, $y(0) = 0$, $y(0.2) = 0.02$, $y(0.4) = 0.0795$, $y(0.6) = 0.1762$ by Milne's method To find $y(0.8)$ and $y(1)$.

Soln.

Here $x_0 = 0$, $y_0 = 0$

$x_1 = 0.2$, $y_1 = 0.02$ and $h = 0.2$

$x_2 = 0.4$, $y_2 = 0.0795$

$x_3 = 0.6$, $y_3 = 0.1762$

$x_4 = 0.8$, $y_4 = ?$

$x_5 = 1$, $y_5 = ?$

By Milne's predictor formula,

$$y_{n+1,P} = y_{n-3} + \frac{4h}{3}[2y'_{n-2} - y'_{n-1} + 2y'_n]$$

$n=3$, $$y_{4,P} = y_0 + \frac{4h}{3}[2y'_1 - y'_2 + 2y'_3]$$

Given $y' = x - y^2$

$y'_1 = x_1 - y_1^2 = 0.2 - (0.02)^2 = 0.1996$

$y'_2 = x_2 - y_2^2 = 0.4 - (0.0795)^2 = 0.3937$

$y'_3 = x_3 - y_3^2 = 0.6 - (0.1762)^2 = 0.5690$

$$\therefore y_{4,P} = 0 + \frac{4(0.2)}{3}[2(0.1996) - 0.3939 + 2(0.569)]$$

$= 0.3049$

$y'_4 = x_4 - y_4^2 = 0.8 - (0.3049)^2 = 0.707$

$$\therefore y_{4,C} = y_2 + \frac{h}{3}[y'_2 + 4y'_3 + y'_4]$$

$$= 0.0795 + \frac{0.2}{3}[0.3939 + 4(0.5690) + 0.707]$$

$= 0.3046$

$\therefore$ corrected value of y at $x = 0.8$ is 0.3046

To find $y(1)$.

$n=4$, $$y_{5,P} = y_1 + \frac{4h}{3}[2y'_2 - y'_3 + 2y'_4]$$

$= 0.02 + \frac{4\times 0.2}{3}[2\times 0.3937 - 0.5690 + 2\times 0.707]$

$y_5 = 0.4553$

$y_5' = x_5 - y_5^2 = 1-(0.4553)^2 = 0.7927$

$y_{5,c} = y_3 + \frac{h}{3}[y_3' + 4y_4' + y_5']$

$= 0.1762 + \frac{0.2}{3}[0.569 + 4(0.707) + 0.7327]$

$= 0.4515$

$\therefore$ corrected value of y at $x=1$ is 0.4515

2]. Given that $\frac{dy}{dx} = \frac{1}{2}(1+x^2)y^2$; $y(0)=1$, $y(0.1)=1.06$, $y(0.2)=1.12$ and $y(0.3)=1.21$. Evaluate $y(0.4)$ and $y(0.5)$ by Milne's predictor corrector method.

Soln.

Given $\frac{dy}{dx} = y' = \frac{1}{2}(1+x^2)y^2$ and $h=0.1$

Here $x_0=0$, $y_0=1$

$x_1=0.1$, $y_1=1.06$

$x_2=0.2$, $y_2=1.12$

$x_3=0.3$, $y_3=1.21$

$x_4=0.4$, $y_4=?$

$x_5=0.5$, $y_5=?$

By Milne's predictor formula, we've

$y_{n+1,P} = y_{n-3} + \frac{4h}{3}[2y'_{n-2} - y'_{n-1} + 2y'_n] \rightarrow (1)$

Put $n=3$,

$y_{4,P} = y_0 + \frac{4h}{3}[2y_1' - y_2' + 2y_3'] \rightarrow (2)$

Now $y_1' = \frac{1}{2}(1+x_1^2)y_1^2 = \frac{1}{2}(1+(0.1)^2)(1.06)^2$

$= 0.5674$

$$y_2' = \frac{1}{2}(1+x_2^2)\,y_2^2 = \frac{1}{2}(1+(0.2)^2)(1.12)^2$$
$$= 0.6523$$
$$y_3' = \frac{1}{2}(1+x_3^2)\,y_3^2 = \frac{1}{2}(1+(0.3)^2)(1.21)^2$$
$$= 0.7979$$
$$(2) \Rightarrow y_{4,p} = 1 + \frac{4(0.1)}{3}\left[2(0.5674) - 0.6523 + 2(0.7979)\right]$$
$$= 1 + \frac{0.4}{3}(2.0783)$$
$$= 1 + 0.2771$$
$$y(0.4) = 1.2771$$

By Milne's corrector formula, we've

$$y_{n+1,c} = y_{n-1} + \frac{h}{3}(y_{n-1}' + 4y_n' + y_{n+1}') \longrightarrow (2)$$

put $n=3$,

$$y_{4,c} = y_2 + \frac{h}{3}(y_2' + 4y_3' + y_4') \longrightarrow (4)$$

Now $$y_4' = \frac{1}{2}(1+x_4^2)\,y_4^2$$
$$= \frac{1}{2}(1+(0.4)^2)(1.2771)^2$$
$$y_4' = 0.9460$$

$$(3) \Rightarrow y_{4,c} = 1.12 + \frac{0.1}{3}(0.653 + 4(0.7979) + 0.946)$$
$$= 1.12 + \frac{0.1}{3}(4.7906)$$
$$= 1.12 + 0.1597$$
$$y(0.4) = 1.2797$$

To find $y(0.5)$

To get y_5, put $n=4$ in (1), we get

$$y_{5,p} = y_1 + \frac{4h}{3}\left[2y_2' - y_3' + 2y_4'\right] \longrightarrow (5)$$

[illegible] 0.7979 + 2(0.946

Now $y_4' = \frac{1}{2}(1+x_4^2) y_4^2$

$= \frac{1}{2}[(1+(0.4)^2](1.2797)^2$

$= \frac{1}{2}(1.8997)$

$= 0.9498$

$(5) \Rightarrow y_{5,P} = 1.06 + \frac{4(0.1)}{3}[2(0.6523) - 0.7979 - 2(0.9498)]$

$= 1.06 + \frac{0.4}{3}(2.4062)$

$= 1.06 + 0.3208$

$y(0.5) = 1.3808$

By Milne's corrector formula, we've

$y_{n+1,C} = y_{n-1} + \frac{h}{3}[y'_{n-1} + 4y'_n + y'_{n+1}]$

Put $n=4$, we get

$y_{5,C} = y_3 + \frac{h}{3}[y_3' + 4y_4' + y_5'] \rightarrow (6)$

Now $y_5' = \frac{1}{2}(1+x_5^2) y_5^2$

$= \frac{1}{2}(1+(0.5)^2)(1.3808)^2$

$= 1.1917$

$(6) \Rightarrow y_{5,C} = 1.21 + \frac{0.1}{3}[0.7979 + 4(0.9498) + 1.1917]$

$= 1.21 + \frac{0.1}{3}(5.7888)$

$= 1.21 + 0.193$

$y(0.5) = 1.403$

3] Given that $\frac{dy}{dx} = 1 + y^2$; $y(0.6) = 0.6841$, $y(0.4) = 0.4228$, $y(0.2) = 0.2027$, $y(0) = 0$, find $y(-0.2)$ using Milne's method.

Soln.

Given $y' = 1 + y^2$ and $h = -0.2$

Here $x_0 = 0.6$, $y_0 = 0.6841$

$x_1 = 0.4$, $y_1 = 0.4228$

$x_2 = 0.2$, $y_2 = 0.2027$

$x_3 = 0$, $y_3 = 0$

$x_4 = -0.2$, $y_4 = ?$

By milne's predictor formula,

$$y_{n+1,P} = y_{n-3} + \frac{4h}{3}[2y'_{n-2} - y'_{n-1} + 2y'_n] \rightarrow (1)$$

Put $n=3$ in (1), we get

$$y_{4,P} = y_0 + \frac{4h}{3}[2y'_1 - y'_2 + 2y'_3] \rightarrow (2)$$

Now $y'_1 = 1 + y_1^2 = 1 + (0.4228)^2$

$= 1.1788$

$y'_2 = 1 + y_2^2 = 1 + (0.2027)^2$

$= 1.0411$

$y'_3 = 1 + y_3^2 = 1 + 0 = 1$

$(2) \Rightarrow y_{4,P} = 0.6841 + \frac{4(-0.2)}{3}[2(1.1788) - 1.0411 + 2(1)]$

$= 0.6841 - 0.2667(3.3165)$

$= 0.6841 - 0.8844$

$= -0.2003$

By milne's corrector formula, we have

$$y_{n+1,c} = y_{n-1} + \frac{h}{3}[y'_{n-1} + 4y'_n + y'_{n+1}]$$

Put $n=3$,

$$y_{4,c} = y_2 + \frac{h}{3}[y'_2 + 4y'_3 + y'_4]$$

Now,

$y'_4 = 1 + y_4^2$

$= 1 + (-0.2003)^2$

$= 1.0401$

$= 0.2027 + \frac{0.2}{3}[1.0411 + 4(1) + 1.0401]$

$= 0.2027 - 0.4054$

$y(-0.2) = -0.2027$

4). Given $y' = 1 - y$ and $y(0) = 0$, find

i). $y(0.1)$ by Euler method ii). $y(0.2)$ by modified Euler method

iii). $y(0.3)$ by Runge Kutta method

iv) $y(0.4)$ by Milne's method.

Soln.:

Given $y' = 1 - y$, $y(0) = 0$

Here $x_0 = 0$, $y_0 = 0$,

$x_1 = 0.1$, $y_1 = ?$ (Euler's method)

$x_2 = 0.2$, $y_2 = ?$ (modified Euler method)

$x_3 = 0.3$, $y_3 = ?$ (R-K method)

$x_4 = 0.4$, $y_4 = ?$ (Milne's method)

i). Euler's method

$$y_{n+1} = y_n + h f(x_n, y_n)$$

$n = 0$ $\quad y_1 = y_0 + h f(x_0, y_0)$

$= 0 + 0.1\, f(0, 0)$

$= 0 + 0.1\,(1 - 0)$

$y(0.1) = 0.1$

Here $x_1 = 0.1$, $y_1 = 0.1$

ii). modified Euler's method

$$y_{n+1} = y_n + h\, f\left(x_n + \frac{h}{2},\ y_n + \frac{h}{2} f(x_n, y_n)\right)$$

$n = 1$, $\quad y_2 = y_1 + h f\left(x_1 + \frac{h}{2},\ y_1 + \frac{h}{2} f(x_1, y_1)\right)$

$= 0.1 + 0.1\, f\left(0.1 + \frac{0.1}{2},\ 0.1 + \frac{0.1}{2} f(0.1, 0.1)\right)$

$y(0.2) = 0.1855$

Here $x_2 = 0.2$, $y_2 = 0.1855$

iii). R-K method

$K_1 = h f(x_2, y_2) = 0.1\,(1 - y_2) = 0.1\,(1 - 0.1855)$

$= 0.8145$

$K_2 = h f\left[x_2 + \frac{h}{2},\ y_2 + \frac{K_1}{2}\right]$

$= 0.1\, f\left(0.2 + \frac{0.1}{2},\ 0.1855 + \frac{0.8145}{2}\right)$

$= 0.1\ F(0.25, 0.6728)$

$= 0.1(1-0.6728)$

$K_2 = 0.0327$

$K_3 = h\ F[x_2 + \frac{h}{2}, y_2 + \frac{K_2}{2}]$

$= 0.1\ F[0.25, 0.1855 + \frac{0.0327}{2}]$

$= 0.1\ F[0.25, 0.2019]$

$= 0.1\ (1-0.2019)$

$K_3 = 0.0798$

$K_4 = hF(x_2 + h, y_2 + K_3) = 0.1\ F(0.2+0.1, 0.1855 + 0.0798)$

$= 0.1\ F(0.3, 0.2653) = 0.1\ (1-0.2653)$

$K_4 = 0.0735$

$\Delta y = \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4) = \frac{1}{6}(0.8145 + 2(0.0327) + 2(0.0798) + 0.0735)$

$= 0.1855$

$y_3 = y_2 + \Delta y = 0.1655 + 0.1855 = 0.3710$

$y(0.3) = 0.3710$

iv). Milne's method

By Milne's Predictor method,

$$y_{n+1,P} = y_{n-3} + \frac{4h}{3}[2y'_{n-2} - y'_{n-1} + 2y'_n] \rightarrow (1)$$

Put $n=3$,

$$y_{4,P} = y_0 + \frac{4h}{3}[2y'_1 - y'_2 + 2y'_3] \rightarrow (2)$$

Now,

$y'_1 = 1 - y_1 = 1 - 0.1 = 0.9$

$y'_2 = 1 - y_2 = 1 - 0.1855 = 0.8145$

$y'_3 = 1 - y_3 = 1 - 0.371 = 0.629$

$(2) \Rightarrow\ y_{4,P} = 0 + \frac{4(0.1)}{3}[2(0.9) - 0.8145 + 2(0.629)]$

$= 0.1333[1.8 - 0.8145 + 1.258]$

$= 0.1333(2.2435)$

$= 0.2991$

By milne's corrector formula,

$$y_{n+1,c} = y_{n-1} + \frac{h}{3}[y'_{n-1} + 4y'_n + y'_{n+1}]$$

put $n=3$,

$$y_{4,c} = y_2 + \frac{h}{3}[y'_2 + 4y'_3 + y'_4]$$

Now

$$y'_4 = 1 - y_4 = 1 - 0.2991 = 0.7009$$

$$y_{4,c} = 0.1855 + \frac{0.1}{3}[0.8145 + 4(0.629) + 0.7009]$$

$$= 0.1855 + \frac{0.1}{3}(2.7502)$$

$$= 0.2772$$

$$y(0.4) = 0.2772$$

5]. using milne's method find $y(4.4)$ given $5xy' + y^2 - 2 = 0$ given $y(4) = 1$, $y(4.1) = 1.0049$, $y(4.2) = 1.0097$ and $y(4.3) = 1.0143$.

Soln.

Given $y' = \frac{2-y^2}{5x}$

Here $x_0 = 4$, $y_0 = 1$

$x_1 = 4.1$, $y_1 = 1.0049$

$x_2 = 4.2$, $y_2 = 1.0097$

$x_3 = 4.3$, $y_3 = 1.0143$

$x_4 = 4.4$, $y_4 = ?$

By milne's predictor formula,

$$y_{n+1,P} = y_{n-3} + \frac{4h}{3}[2y'_{n-2} - y'_{n-1} + 2y'_n]$$

put $n=3$,

$$y_{4,P} = y_0 + \frac{4h}{3}[2y'_1 - y'_2 + 2y'_3]$$

Now, $y'_1 = \frac{2-y_1^2}{5x_1} = \frac{2-(1.0049)^2}{5(4.1)} = 0.0493$

$$y'_2 = \frac{2-y_2^2}{5x_2} = \frac{2-(1.0097)^2}{5(4.2)} = 0.0467$$

$$y_3' = \frac{2-y_3^2}{5x_3} = \frac{2-(1.0143)^2}{5(4.3)} = 0.0452$$

$$\therefore y_{4,P} = 1 + \frac{4(0.1)}{3}[2(0.0493) - 0.0467 + 2(0.0452)]$$
$$= 1.019$$

Now, $$y_4' = \frac{2-y_4^2}{5x_4} = \frac{2-(1.019)^2}{5(4.4)} = 0.0437$$

By Milne's corrector formula,

$$y_{n+1,C} = y_{n-1} + \frac{h}{3}[y_{n-1}' + 4y_n' + y_{n+1}']$$

For $n=3$,

$$y_{4,C} = y_2 + \frac{h}{3}[y_2' + 4y_3' + y_4']$$

$$= 1.0097 + \frac{0.1}{3}[0.0467 + 4(0.0452) + 0.0437]$$

$$= 1.0187$$

$$\therefore y(4.4) = 1.0187$$