STOKE'S THEOREM :

If $\vec{F}$ is any continuous differentiable vector function and S is a surface enclosed by a curve c then,

$$\boxed{\int_C \vec{F}.d\vec{r} = \iint_S \nabla \times \vec{F}.\hat{n}\, ds}$$

Where $\hat{n}$ is the unit normal vector at any point of S.

Note :

1. If $\vec{F}$ is irrotational, $\nabla \times \vec{F} = 0$

$$\int_C \vec{F}.d\vec{r} = \iint_S \nabla \times \vec{F}.\hat{n}\, ds = 0$$ & hence $\vec{F}$ is conservative.

2. Let $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$

$$\int_C P\,dx + Q\,dy + R\,dz = \iint_S \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) dy\,dz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) dz\,dx + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

PROBLEMS :

1) Verify stoke's theorem for a vector field defined by $\vec{F} = (x^2 - y^2)\vec{i} + 2xy\,\vec{j}$ in the rectangular region in the XOY plane bounded by the lines $x=0$, $x=a$, $y=0$ and $y=b$.

Soln :

By Stoke's theorem,

$$\int_C \vec{F}.d\vec{r} = \iint_S \nabla \times \vec{F}.\hat{n}\, ds$$

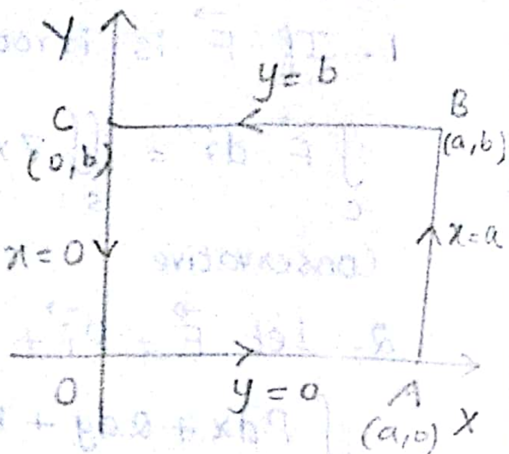
RHS :

Given : $\vec{F} = (x^2 - y^2)\vec{i} + 2xy\vec{j}$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2 - y^2 & 2xy & 0 \end{vmatrix}$$

$$= \vec{i}(0) - \vec{j}(0) + \vec{k}(2y + 2y) = 4y\vec{k}$$

Here the surface S denotes the rectangle OABC and the unit outward normal vector is $\vec{k}$.



i.e., $\hat{n} = \vec{k}$

$\therefore$ Curl $\vec{F} \cdot \hat{n}\, ds$

$$= 4y\vec{k} \cdot \vec{k}\, dx\, dy$$

$$= 4y\, dx\, dy$$

$$\therefore \iint \nabla \times \vec{F} \cdot \hat{n}\, ds = \iint_S 4y\, dx\, dy$$

$$= \int_0^b \int_0^a 4y\, dx\, dy$$

$$= 4\int_0^b [x]_0^a\, y\, dy$$

$$= 4a\left[\frac{y^2}{2}\right]_0^b$$

$$= \frac{4ab^2}{2}$$

$$= 2ab^2 \longrightarrow (1)$$

LHS :

$$\vec{F} = (x^2-y^2)\,\vec{i} + 2xy\,\vec{j}$$

$$d\vec{r} = dx\,\vec{i} + dy\,\vec{j}$$

$$\vec{F}\cdot d\vec{r} = (x^2-y^2)\,dx + 2xy\,dy$$

$$\int_C \vec{F}\cdot d\vec{r} = \int_C \left[(x^2-y^2)\,dx + 2xy\,dy\right]$$

$$= \int_{OA} + \int_{AB} + \int_{BC} + \int_{CO}$$

Along OA $(y=0)$:

$$\int_{OA} (x^2-y^2)\,dx + 2xy\,dy = \int_0^a x^2\,dx \qquad \left[\because y=0 \Rightarrow dy=0 \text{ Along OA, } x \to 0 \text{ to } a\right]$$

$$= \left[\frac{x^3}{3}\right]_0^a$$

$$= \frac{a^3}{3}$$

Along AB $(x=a)$:

$$\int_{AB} (x^2-y^2)\,dx + 2xy\,dy = \int_0^b 2ay\,dy \qquad \left[\because x=a \Rightarrow dx=0 \text{ Along AB } y \to 0 \text{ to } b\right]$$

$$= 2a\left[\frac{y^2}{2}\right]_0^b$$

$$= ab^2$$

Along BC $(y=b)$:

$$\int_{BC} (x^2-y^2)\,dx + 2xy\,dy = \int_a^0 (x^2-b^2)\,dx \qquad \left[\because y=b \Rightarrow dy=0 \text{ Along BC } x \to a \text{ to } 0\right]$$

$$= \left[\frac{x^3}{3} - b^2x\right]_a^0$$

$$= -\frac{a^3}{3} + ab^2$$

Along $CO\ (x=0)$:

$$\int_{CO} (x^2-y^2)\,dx + 2xy\,dy = \int_{CO} 0 + 0 \qquad (\because x=0,\ dx=0)$$

Hence
$$\int_C \vec{F}\cdot d\vec{r} = \int_{OA} + \int_{AB} + \int_{BC} + \int_{CO}$$

$$= \frac{a^3}{3} + ab^2 - \frac{a^3}{3} + ab^2 + 0$$

$$= 2ab^2 \rightarrow (2)$$

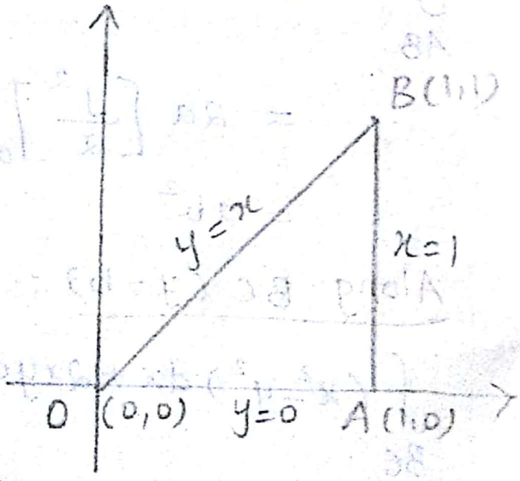
From (1) & (2),
$$\int_C \vec{F}\cdot d\vec{r} = \iint_S \nabla\times\vec{F}\cdot\hat{n}\,ds$$

Hence Stoke's theorem is verified.

(2) Evaluate $\int_C \vec{F}\cdot d\vec{r}$ by Stoke's theorem where $\vec{F} = y^2\vec{i} + x^2\vec{j} - (x+z)\vec{k}$ and C is the boundary of the triangle with vertices at $(0,0,0)$, $(1,0,0)$, $(1,1,0)$.

Soln:

By Stoke's theorem,
$$\int_C \vec{F}\cdot d\vec{r} = \iint_S \nabla\times\vec{F}\cdot\hat{n}\,ds$$



Since z coordinate is zero in all the three vertices of the given triangle, the triangle lies on the xy plane.

$$\vec{F} = y^2\vec{i} + x^2\vec{j} - (x+z)\vec{k}$$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^2 & x^2 & -(x+z) \end{vmatrix}$$

$$= \vec{i}(0) - \vec{j}(-1-0) + \vec{k}(2x-2y)$$

$$= \vec{j} + 2(x-y)\vec{k}$$

Since the triangle lies on XOY plane and hence the unit vector normal to the surface OAB is $\vec{k}$.

i.e., $\hat{n} = \vec{k}$

$$\nabla \times \vec{F} \cdot \hat{n} = [\vec{j} + 2(x-y)\vec{k}] \cdot \vec{k} = 2(x-y)$$

$$\therefore \int_C \vec{F} \cdot d\vec{r} = \iint_S \nabla \times \vec{F} \cdot \hat{n} \, ds$$

$$= \iint_S 2(x-y)\, dx\, dy \quad (\because S \text{ lies on } XY \text{ plane } ds = dx\, dy)$$

$$= 2\int_0^1 \int_y^1 (x-y)\, dx\, dy \quad [x \to y \text{ to } 1,\ y \to 0 \text{ to } 1]$$

$$= 2\int_0^1 \left[\frac{x^2}{2} - xy\right]_y^1 dy$$

$$= 2\int_0^1 \left[\frac{1}{2} - y - \frac{y^2}{2} + y^2\right] dy$$

$$= 2\left[\frac{1}{2}y - \frac{y^2}{2} - \frac{y^3}{6} + \frac{y^3}{3}\right]_0^1$$

$$= 2\left(\frac{1}{2} - \frac{1}{2} - \frac{1}{6} + \frac{1}{3}\right)$$

$$= \frac{1}{3}.$$