GAUSS DIVERGENCE THEOREM :

If $\vec{F}$ is a vector point function, finite and differentiable in a region R bounded by a closed Surface S, then the surface integral of the normal component of $\vec{F}$ taken over S is equal to the integral of divergence of $\vec{F}$ taken over V.

$$\boxed{\iint_S \vec{F} \cdot \hat{n}\, ds = \iiint_V \nabla \cdot \vec{F}\, dv}$$

where $\hat{n}$ is the unit vector in the positive (outward drawn) normal to S.

Problems :

① Verify Gauss divergence theorem for $\vec{F} = 4xz\,\vec{i} - y^2\,\vec{j} + yz\,\vec{k}$ over the cube $x=0, x=1, y=0, y=1, z=0, z=1$.

Soln: By Gauss divergence theorem,

$$\iint_S \vec{F} \cdot \hat{n}\, ds = \iiint_V \nabla \cdot \vec{F}\, dv$$

RHS :

$$\nabla \cdot \vec{F} = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right) \cdot (4xz\,\vec{i} - y^2\,\vec{j} + yz\,\vec{k})$$

$$= \frac{\partial}{\partial x}(4xz) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(yz)$$

$$= 4z - 2y + y$$

$$= 4z - y$$

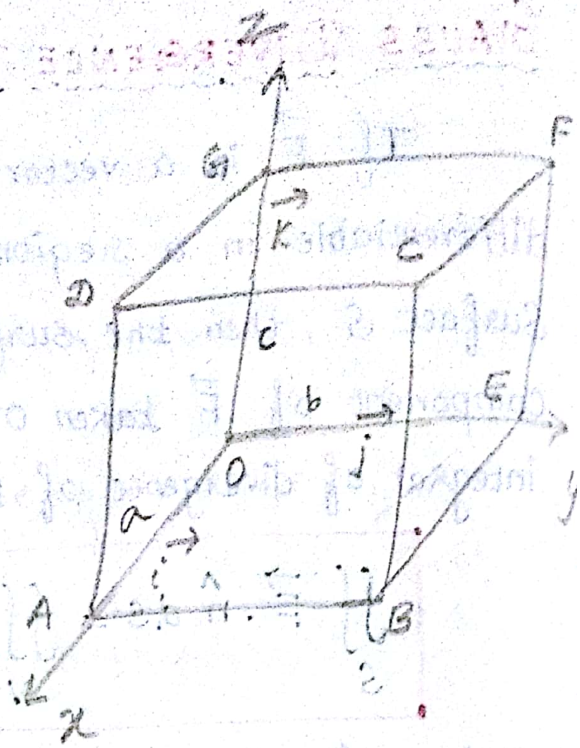
$$\iiint_V \nabla \cdot \vec{F}\, dv = \int_0^1 \int_0^1 \int_0^1 (4z - y)\, dx\, dy\, dz = \frac{3}{2} \longrightarrow ①$$

LHS :

$$\iint_S \vec{F} \cdot \hat{n} \, ds$$

$$= \iint_{S_1} + \iint_{S_2} + \iint_{S_3} +$$

$$\iint_{S_4} + \iint_{S_5} + \iint_{S_6}$$



Surface	$\hat{n}$	ds	Face equation
S_1 - ABCD	$\vec{i}$	$dy\,dz$	$x=1$
S_2 - OEFG	$-\vec{i}$	$dy\,dz$	$x=0$
S_3 - BCEF	$\vec{j}$	$dx\,dz$	$y=1$
S_4 - OADG	$-\vec{j}$	$dx\,dz$	$y=0$
S_5 - DCGF	$\vec{K}$	$dx\,dy$	$z=1$
S_6 - OABE	$-\vec{K}$	$dx\,dy$	$z=0$

$$\iint_{S_1} \vec{F} \cdot \hat{n} \, ds = \iint_{ABCD} (4xz\vec{i} - y^2\vec{j} + yz\vec{K}) \cdot \vec{i} \, dy\,dz$$

$$= \int_0^1\int_0^1 4xz \, dy\,dz$$

$$= \int_0^1\int_0^1 4z \, dy\,dz \quad (\because x=1)$$

$$= 2$$

$$\iint_{S_2} \vec{F} \cdot \hat{n} \, ds = \iint_{OEFG} (4xz\vec{i} - y^2\vec{j} + yz\vec{K}) \cdot (-\vec{i}) \, dy\,dz$$

$$= \int_0^1\int_0^1 (-4xz)\, dy\, dz = 0 \quad (\because x = 0)$$

$$\iint_{S_3} \vec{F}\cdot\hat{n}\, ds = \iint_{BCEF} (4xz\vec{i} - y^2\vec{j} + yz\vec{k})\cdot\vec{j}\, dx\, dz$$

$$= \int_0^1\int_0^1 (-y^2)\, dx\, dz \quad (\text{Here } y = 1)$$

$$= \int_0^1\int_0^1 -dx\, dz = -1$$

$$\iint_{S_4} \vec{F}\cdot\hat{n}\, ds = \iint_{OADG} (4xz\vec{i} - y^2\vec{j} + yz\vec{k})\cdot(-\vec{j})\, dx\, dz$$

$$= \int_0^1\int_0^1 y^2\, dx\, dz = 0 \quad (\because y = 0)$$

$$\iint_{S_5} \vec{F}\cdot\hat{n}\, ds = \iint_{DCGF} (4xz\vec{i} - y^2\vec{j} + yz\vec{k})\cdot\vec{k}\, dx\, dy$$

$$= \int_0^1\int_0^1 yz\, dx\, dy = \int_0^1\int_0^1 y\, dx\, dy \quad (\because z = 1)$$

$$= \int_0^1 y\left[x\right]_0^1 dy = \int_0^1 y\, dy = \left[\frac{y^2}{2}\right]_0^1 = \frac{1}{2}$$

$$\iint_{S_6} \vec{F}\cdot\hat{n}\, ds = \iint_{OABE} (-yz)\, dx\, dy = 0 \quad (\because z = 0)$$

$$\iint_S \vec{F}\cdot\hat{n}\, ds = \iint_{S_1} + \iint_{S_2} + \iint_{S_3} + \iint_{S_4} + \iint_{S_5} + \iint_{S_6}$$

$$= 2 + 0 + (-1) + 0 + 1/2 + 0$$

$$= 3/2 \longrightarrow (2)$$

From (1) & (2),

$$\iint_S \vec{F}\cdot\hat{n}\, ds = \iiint_V \nabla\cdot\vec{F}\, dv$$

② Verify divergence theorem for $\vec{F} = (x^2 - yz)\vec{i} + (y^2 - zx)\vec{j} + (z^2 - xy)\vec{k}$ taken over the rectangular Parallelopiped $0 \le x \le a,\ 0 \le y \le b,\ 0 \le z \le c$.

(or)

Verify Gauss divergence theorem for $\vec{F} = (x^2 - yz)\vec{i} + (y^2 - zx)\vec{j} + (z^2 - xy)\vec{k}$ and s is the surface of the rectangular parallelopiped bounded by $x=0,\ x=a,\ y=0,\ y=b,\ z=0,\ z=c$.

Soln :

By Gauss-divergence theorem,

$$\iint_S \vec{F} \cdot \hat{n}\, ds = \iiint_V \nabla \cdot \vec{F}\, dv.$$

RHS :

$$\vec{F} = (x^2 - yz)\vec{i} + (y^2 - zx)\vec{j} + (z^2 - xy)\vec{k}$$

$$\nabla \cdot \vec{F} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \cdot \left((x^2 - yz)\vec{i} + (y^2 - zx)\vec{j} + (z^2 - xy)\vec{k}\right)$$

$$= \frac{\partial}{\partial x}(x^2 - yz) + \frac{\partial}{\partial y}(y^2 - zx) + \frac{\partial}{\partial z}(z^2 - xy)$$

$$= (2x + 2y + 2z)$$

$$= 2(x + y + z).$$

$$\iiint_V \nabla \cdot \vec{F}\, dv = \int_0^a \int_0^b \int_0^c 2(x + y + z)\, dx\, dy\, dz$$

$$= 2\int_0^a \int_0^b \left[xz + yz + \frac{z^2}{2}\right]_0^c dx\, dy$$

$$= 2\int_0^a \int_0^b \left(xc + yc + \frac{c^2}{2}\right) dx\, dy$$

$$= 2\int_0^a \left[xyc + \frac{y^2}{2}c + \frac{c^2}{2}\cdot y \right]_0^b dx$$

$$= 2\int_0^a \left(xbc + \frac{b^2c}{2} + \frac{c^2b}{2} \right) dx$$

$$= 2\left[\frac{x^2}{2}bc + \frac{b^2c}{2}x + \frac{c^2b}{2}x \right]_0^a$$

$$= 2\left[\frac{a^2bc}{2} + \frac{ab^2c}{2} + \frac{abc^2}{2} \right]$$

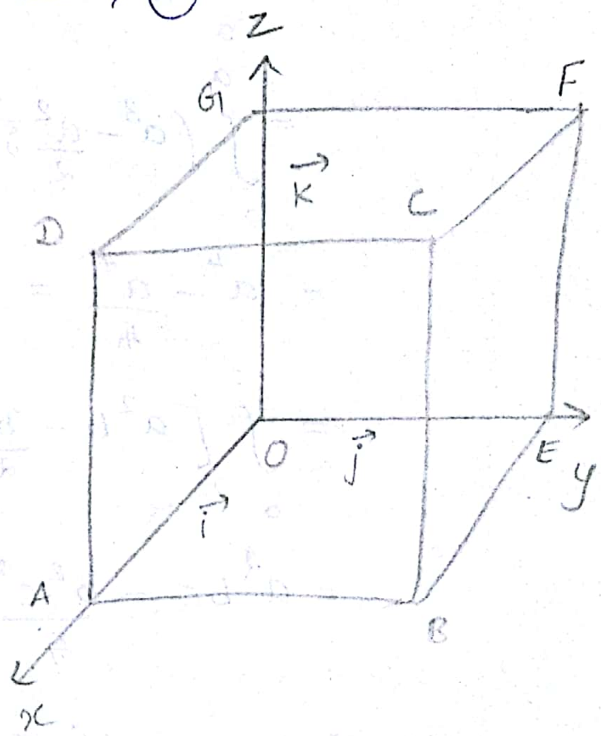
$$= abc(a+b+c) \longrightarrow (1)$$

LHS :

$$\iint_S \vec{F}\cdot\hat{n}\, ds$$

$$= \iint_{S_1} + \iint_{S_2} + \iint_{S_3} +$$

$$\iint_{S_4} + \iint_{S_5} + \iint_{S_6}$$



Surface	$\hat{n}$	ds	Face equation
S_1 - ABCD	$\vec{i}$	dy dz	$x = a$
S_2 - OEFG	$-\vec{i}$	dy dz	$x = 0$
S_3 - BCEF	$\vec{j}$	dx dz	$y = b$
S_4 - OADG	$-\vec{j}$	dx dz	$y = 0$
S_5 - DCGF	$\vec{K}$	dx dy	$z = c$
S_6 - OABE	$-\vec{K}$	dx dy	$z = 0$

$$\iint_{S_1} \vec{F}\cdot\hat{n}\, ds = \iint_{ABCD} \left[(x^2-yz)\vec{i} + (y^2-zx)\vec{j} + (z^2-xy)\vec{k}\right]\cdot\vec{i}\; dy\, dz$$

$$= \int_0^c\int_0^b (x^2-yz)\, dy\, dz$$

$$= \int_0^c\int_0^b (a^2-yz)\, dy\, dz \qquad (\because x=a)$$

$$= \int_0^c \left[a^2 y - \frac{y^2}{2} z\right]_0^b dz$$

$$= \int_0^a \left(a^3 - \frac{a^2}{2} z\right) dz = \left[a^3 z - \frac{a^2}{2}\frac{z^2}{2}\right]_0^a$$

$$= a^4 - \frac{a^4}{4} = \frac{4a^4 - a^4}{4} = \frac{3a^4}{4}$$

$$= \int_0^c \left[a^2 b - \frac{b^2}{2} z\right] dz = \left[a^2 b z - \frac{b^2}{2}\frac{z^2}{2}\right]_0^c$$

$$= a^2 bc - \frac{b^2 c^2}{4}$$

$$\iint_{S_2} \vec{F}\cdot\hat{n}\, ds = \iint_{OEFG} \left[(x^2-yz)\vec{i} + (y^2-zx)\vec{j} + (z^2-xy)\vec{k}\right]\cdot(-\vec{i})\, dy\, dz$$

$$= \int_0^c\int_0^b -(x^2-yz)\, dy\, dz$$

$$= \int_0^c\int_0^b yz\, dy\, dz \qquad (\because x=0)$$

$$= \int_0^c \left[\frac{y^2}{2}\right]_0^b z\, dz = \frac{b^2}{2}\left[\frac{z^2}{2}\right]_0^c = \frac{b^2c^2}{4}$$

$$\iint_{S_3} \vec{F}\cdot\hat{n}\, ds = \iint_{BCEF} \left[(x^2-yz)\vec{i} + (y^2-zx)\vec{j} + (z^2-xy)\vec{k}\right]\cdot\vec{j}\, dx\, dz$$

$$= \int_0^a\int_0^c (y^2-zx)\, dx\, dz$$

$$= \int_0^a\int_0^c (b^2-zx)\, dx\, dz \qquad (\because y=b)$$

$$= \int_0^a \left[b^2x - \frac{zx^2}{2}\right]_0^c dz$$

$$= \int_0^a \left[b^2c - \frac{zc^2}{2}\right] dz = \left[b^2cz - \frac{z^2c^2}{4}\right]_0^a$$

$$= ab^2c - \frac{a^2c^2}{4}$$

$$\iint_{S_4} \vec{F}\cdot\hat{n}\, ds = \iint_{OADG} \left[(x^2-yz)\vec{i} + (y^2-zx)\vec{j} + (z^2-xy)\vec{k}\right]\cdot(-\vec{j})\, dx\, dz$$

$$= \int_0^a\int_0^c -(y^2-zx)\, dx\, dz$$

$$= \int_0^a\int_0^c zx\, dx\, dz \qquad (\because y=0)$$

$$= \int_0^a \left[\frac{x^2}{2}\right]_0^c z\, dz$$

$$= \frac{c^2}{2}\left[\frac{z^2}{2}\right]_0^a$$

$$= \frac{a^2c^2}{4}$$

$$\iint_{S_5} \vec{F}\cdot\hat{n}\, ds = \iint_{DCGF} [(x^2-yz)\vec{i}+(y^2-zx)\vec{j}+(z^2-xy)\vec{k}] \cdot \vec{k}\, dx\, dy$$

$$= \int_0^a\int_0^b (z^2-xy)\, dx\, dy$$

$$= \int_0^a\int_0^b (c^2-xy)\, dx\, dy \qquad (\because z=c)$$

$$= \int_0^a \left[c^2x - \frac{x^2}{2}y\right]_0^b dy$$

$$= \int_0^a \left(bc^2 - \frac{b^2}{2}y\right) dy = \left(bc^2y - \frac{b^2y^2}{4}\right)_0^a$$

$$= abc^2 - \frac{a^2b^2}{4}$$

$$\iint_{S_6} \vec{F}\cdot\hat{n}\, ds = \iint_{OABE} [(x^2-yz)\vec{i}+(y^2-zx)\vec{j}+(z^2-xy)\vec{k}] \cdot (-\vec{k})\, dx\, dy$$

$$= \int_0^a\int_0^b -(z^2-xy)\, dx\, dy$$

$$= \int_0^a\int_0^b xy\, dx\, dy \qquad (\because z=0)$$

$$= \int_0^a y\, dy \left[\frac{x^2}{2}\right]_0^b = \frac{b^2}{2}\left[\frac{y^2}{2}\right]_0^a$$

$$= \frac{a^2b^2}{4}$$

$$\therefore \iint_S \vec{F}\cdot\hat{n}\, ds = a^2bc - \cancel{\frac{b^2c^2}{4}} + \cancel{\frac{b^2c^2}{4}} + ab^2c - \cancel{\frac{a^2c^2}{4}} + \cancel{\frac{a^2c^2}{4}} + abc^2 - \cancel{\frac{a^2b^2}{4}} + \cancel{\frac{a^2b^2}{4}}$$

$$= abc(a+b+c) \longrightarrow (2)$$

Scanned by CamScanner

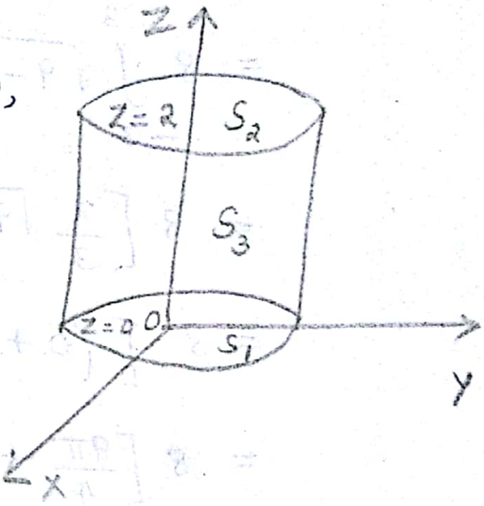
From ① and ②,

$$\iint_S \vec{F} \cdot \hat{n}\, ds = \iiint_V \nabla \cdot \vec{F}\, dv.$$

③ Verify Gauss divergence theorem for the function $\vec{F} = y\vec{i} + x\vec{j} + z^2\vec{K}$ over the cylindrical region bounded by $x^2+y^2=9$, $z=0$ and $z=2$.

Soln:

By Gauss divergence theorem,



$$\iint_S \vec{F} \cdot \hat{n}\, ds = \iiint_V \nabla \cdot \vec{F}\, dv$$

RHS:

Given : $\vec{F} = y\vec{i} + x\vec{j} + z^2\vec{k}$

$$\nabla \cdot \vec{F} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \cdot (y\vec{i} + x\vec{j} + z^2\vec{k})$$

$$= \frac{\partial}{\partial x}(y) + \frac{\partial}{\partial y}(x) + \frac{\partial}{\partial z}(z^2) = 0+0+2z$$

$$= 2z$$

The region is bounded by $z=0$ & $z=2$

$$x^2+y^2=9$$

$$y^2 = 9-x^2$$

$$y = \pm\sqrt{9-x^2}$$

$$\therefore \iiint_V \nabla \cdot \vec{F}\, dv = \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_0^2 2z\, dz\, dy\, dx$$

$$= \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} 2\left(\frac{z^2}{2}\right)_0^2 dy\, dx$$

$$= \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} (z^2)_0^2 \, dy \, dx$$

$$= 4 \int_{-3}^{3} [y]_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \, dx$$

$$= 4 \int_{-3}^{3} 2\sqrt{9-x^2} \, dx$$

$$= 8 \int_{-3}^{3} \sqrt{9-x^2} \, dx$$

$$= 8 \left[\frac{x}{3}\sqrt{9-x^2} + \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) \right]_{-3}^{3}$$

$$= 8 \left[\left(0 + \frac{9}{2}\cdot\frac{\pi}{2}\right) - \left(0 - \frac{9}{2}\cdot\frac{\pi}{2}\right) \right]$$

$$= 8\left[\frac{9\pi}{4} + \frac{9\pi}{4}\right] = 8\left(\frac{9\pi}{2}\right) = 36\pi \longrightarrow (1)$$

LHS:

$$\iint_S \vec{F}\cdot\hat{n}\, ds = \iint_{S_1} + \iint_{S_2} + \iint_{S_3}$$

$$\iint_{S_1} \vec{F}\cdot\hat{n}\, ds = \iint_{S_1} \vec{F}\cdot(-\vec{k})\, dx\,dy$$

$$= \iint_{S_1} (y\vec{i} + x\vec{j} + z^2\vec{k})\cdot(-\vec{k})\, dx\,dy$$

$$= \iint_{S_1} -z^2 \, dx\,dy$$

$$= 0 \qquad (\because z = 0 \text{ on } S_1)$$

$$\iint_{S_2} \vec{F}\cdot\hat{n}\, ds = \iint_{S_2} \vec{F}\cdot\vec{k}\, dx\,dy$$

$$= \iint_{S_2} (y\vec{i} + x\vec{j} + z^2\vec{k})\cdot\vec{k}\, dx\,dy$$

Scanned by CamScanner

$$= \iint_{S_2} z^2 \, dx\, dy$$

$$= \iint_{S_2} 2^2 \, dx\, dy \quad (\because z = 2 \text{ on } S_2)$$

$$= 4(9\pi) \quad (\because S_2 = \text{Area of } S_2 = \pi r^2 = \pi(3)^2 = 9\pi)$$

$$= 36\pi$$

To find

$$\iint_{S_3} \vec{F} \cdot \hat{n}\, ds.$$

Given: $x^2 + y^2 = 9$

$$\varphi = x^2 + y^2 - 9$$

$$\nabla\varphi = 2x\vec{i} + 2y\vec{j} = 2(x\vec{i} + y\vec{j})$$

$$|\nabla\varphi| = \sqrt{4x^2 + 4y^2} = 2\sqrt{x^2+y^2} = 2\sqrt{9} = 2\times 3 = 6.$$

$$\hat{n} = \frac{\nabla\varphi}{|\nabla\varphi|} = \frac{2(x\vec{i} + y\vec{j})}{6} = \frac{x\vec{i} + y\vec{j}}{3}$$

$$\vec{F} \cdot \hat{n} = (y\vec{i} + x\vec{j} + z^2\vec{k}) \cdot \frac{1}{3}(x\vec{i} + y\vec{j})$$

$$= \frac{1}{3}xy + \frac{1}{3}xy = \frac{2}{3}xy$$

Projecting the surface S_3 on the yz plane

then $ds = \dfrac{dy\,dz}{|\hat{n}\cdot\vec{i}|}$

$$\hat{n}\cdot\vec{i} = \frac{1}{3}(x\vec{i} + y\vec{j})\cdot\vec{i} = \frac{1}{3}x$$

$$ds = \frac{dy\,dz}{\frac{1}{3}x} = \frac{3}{x}\,dy\,dz$$

$$\iint_{S_3} \vec{F}\cdot\hat{n}\, ds = \int_0^2 \int_{-3}^{3} \frac{2}{3}xy\,\frac{3}{x}\,dy\,dz$$

$$(\because x^2 + y^2 = 9$$
$$x = 0 \Rightarrow y^2 = 9$$
$$y = \pm 3)$$

$$= \int_0^2 \int_{-3}^{3} 2y\,dy\,dz = 0$$

$$\iint_S = \iint_{S_1} + \iint_{S_2} + \iint_{S_3}$$

$$= 0 + 36\pi + 0$$

$$= 36\pi \longrightarrow (2)$$

From (1) & (2), LHS = RHS.

Hence Gauss divergence theorem is verified.