UNIT - II

VECTOR CALCULUS

Scalar Quantity:

A scalar quantity is that which has magnitude and is not related to any direction.

Vector Quantity:

A vector quantity is that which has both magnitude and direction.

Scalar point function:

If corresponding to each point P of a region R there corresponds a scalar denoted by $\varphi(P)$ or $\varphi(x, y, z)$ then φ is said to be a scalar point function for the region R.

Example: The temperature $\varphi(P)$ at any point P of a body occupying a certain region is a scalar point function.

Vector point function:

If corresponding to each point P of a region R, there corresponds a vector denoted by F(P), then F is said to be a vector point function for the region R.

Example: The acceleration F(P) of a particle at any time t occupying the position P in a certain region is a vector point function.

Vector Differential Operator:

The vector differential operator ∇ is defined as,

$$\boxed{\nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z} = \sum \vec{i}\frac{\partial}{\partial x}}$$

Gradient of a Scalar point function:

Let $\varphi(x, y, z)$ be a scalar point function and is continuously differentiable then the vector

$$\boxed{\begin{aligned}\nabla\varphi &= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\varphi \\ &= \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}\end{aligned}}$$

is called the gradient of φ and is written as grad φ.

i.e., $\boxed{\text{grad } \varphi = \nabla\varphi}$

Note:

1. $\nabla\varphi$ defines a vector field.
2. $\nabla\varphi \neq \varphi\nabla$. There will be no '.' or '×' between φ and ∇.

Properties of Gradient:

1. If f and g are two scalar point functions then,

$$\boxed{\begin{aligned}&\nabla(f \pm g) = \nabla f \pm \nabla g \\ \text{(or)}\quad &\text{grad}(f \pm g) = \text{grad } f \pm \text{grad } g\end{aligned}}$$

(2) If f and g are two scalar point functions then,

$$\boxed{\begin{aligned}&\nabla(fg) = f\nabla g + g\nabla f\\ \text{(or)}\quad &\text{grad}\,(fg) = f(\text{grad}\,g) + g(\text{grad}\,f)\end{aligned}}$$

(3) If f and g are two scalar point functions then,

$$\boxed{\begin{aligned}&\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2} \quad \text{where } g \neq 0\\ \text{(or)}\quad &\text{grad}\left(\frac{f}{g}\right) = \frac{g(\text{grad}\,f) - f(\text{grad}\,g)}{g^2}\end{aligned}}$$

(4) Gradient of a constant is zero.

i.e., $\boxed{\nabla\varphi = 0}$

Problems :

(1) Find grad φ where $\varphi = x^2+y^2+z^2$.

Soln :

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$= \vec{i}\frac{\partial}{\partial x}(x^2+y^2+z^2) + \vec{j}\frac{\partial}{\partial y}(x^2+y^2+z^2) + \vec{k}\frac{\partial}{\partial z}(x^2+y^2+z^2)$$

$$= \vec{i}(2x) + \vec{j}(2y) + \vec{k}(2z)$$

$$\nabla\varphi = 2x\vec{i} + 2y\vec{j} + 2z\vec{k}$$

(2) Find grad φ if $\varphi = xyz$ at $(1,1,1)$

Soln:

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$\nabla\varphi = \vec{i}\frac{\partial}{\partial x}(xyz) + \vec{j}\frac{\partial}{\partial y}(xyz) + \vec{k}\frac{\partial}{\partial z}(xyz)$$

$$= \vec{i}\,yz + \vec{j}(xz) + \vec{k}(xy)$$

$$\boxed{\nabla\varphi_{(1,1,1)} = \vec{i}+\vec{j}+\vec{k}.}$$

③ Find grad φ where $\varphi = 3x^2y - y^3z^2$ at $(1,1,1)$.

Soln:

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$= \vec{i}\frac{\partial}{\partial x}(3x^2y - y^3z^2) + \vec{j}\frac{\partial}{\partial y}(3x^2y - y^3z^2) + \vec{k}\frac{\partial}{\partial z}(3x^2y - y^3z^2)$$

$$= \vec{i}(6xy) + \vec{j}\cdot(3x^2 - 3y^2z^2) + \vec{k}(-2y^3z)$$

$$\boxed{\nabla\varphi_{(1,1,1)} = 6\vec{i} - 2\vec{k}}$$

④ If $\varphi = \log(x^2+y^2+z^2)$ find $\nabla\varphi$.

Soln:

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$= \vec{i}\frac{\partial}{\partial x}[\log(x^2+y^2+z^2)] + \vec{j}\frac{\partial}{\partial y}[\log(x^2+y^2+z^2)] + \vec{k}\frac{\partial}{\partial z}[\log(x^2+y^2+z^2)]$$

$$= \vec{i}\frac{1}{x^2+y^2+z^2}(2x) + \vec{j}\frac{1}{x^2+y^2+z^2}(2y) + \vec{k}\frac{1}{x^2+y^2+z^2}(2z)$$

$$= \frac{2}{x^2+y^2+z^2}(x\vec{i} + y\vec{j} + z\vec{k})$$

$$\boxed{\nabla\varphi = \frac{2\vec{r}}{x^2+y^2+z^2}}$$

(5) Find $\nabla(\log r)$

Soln:

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$\nabla(\log r) = \vec{i}\frac{\partial}{\partial x}(\log r) + \vec{j}\frac{\partial}{\partial y}(\log r) + \vec{k}\frac{\partial}{\partial z}(\log r)$$

$$= \vec{i}\frac{1}{r}\frac{\partial r}{\partial x} + \vec{j}\frac{1}{r}\frac{\partial r}{\partial y} + \vec{k}\cdot\frac{1}{r}\frac{\partial r}{\partial z} \rightarrow (1)$$

Since $r = \sqrt{x^2+y^2+z^2}$

$$\frac{\partial r}{\partial x} = \frac{1}{2\sqrt{x^2+y^2+z^2}}\cdot 2x = \frac{x}{\sqrt{x^2+y^2+z^2}} = \frac{x}{r}$$

Similarly $\frac{\partial r}{\partial y} = \frac{y}{r}$ & $\frac{\partial r}{\partial z} = \frac{z}{r}$

Subs these values in (1),

$$\nabla(\log r) = \vec{i}\frac{1}{r}\left(\frac{x}{r}\right) + \vec{j}\cdot\frac{1}{r}\frac{y}{r} + \vec{k}\cdot\frac{1}{r}\cdot\frac{z}{r}$$

$$= \vec{i}\frac{x}{r^2} + \vec{j}\frac{y}{r^2} + \vec{k}\frac{z}{r^2}$$

$$= \frac{1}{r^2}(x\vec{i} + y\vec{j} + z\vec{k})$$

$$\boxed{\nabla\cdot(\log r) = \frac{\vec{r}}{r^2}}$$

(6) If $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ such that $|\vec{r}| = r$,

Prove that

(i) $\nabla r = \frac{\vec{r}}{r} = \hat{r}$

(ii) $\nabla\left(\frac{1}{r}\right) = \frac{-\vec{r}}{r^3} = \frac{-\hat{r}}{r^2}$

(iii) $\nabla r^n = n r^{n-2} \vec{r}$

(iv) $\nabla f(r) = f'(r) \nabla r$.

(v) $\nabla f(r) \times \vec{r} = 0$

(vi) If $\nabla\varphi$ is Solenoidal find $\nabla^2\varphi$.

Soln :

(i) Given : $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$$|\vec{r}| = r = \sqrt{x^2+y^2+z^2}$$

$$r^2 = x^2+y^2+z^2$$

$$2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$$

$$2r\frac{\partial r}{\partial y} = 2y \Rightarrow \frac{\partial r}{\partial y} = \frac{y}{r}$$

$$2r\frac{\partial r}{\partial z} = 2z \Rightarrow \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\nabla r = \vec{i}\frac{\partial r}{\partial x} + \vec{j}\frac{\partial r}{\partial y} + \vec{k}\frac{\partial r}{\partial z}$$

$$= \vec{i}\frac{x}{r} + \vec{j}\frac{y}{r} + \vec{k}\frac{z}{r}$$

$$= \frac{1}{r}(x\vec{i} + y\vec{j} + z\vec{k})$$

$$\boxed{\nabla r = \frac{\vec{r}}{r} = \hat{r}}$$

(ii) $\nabla\left(\frac{1}{r}\right) = \vec{i}\frac{\partial}{\partial x}\left(\frac{1}{r}\right) + \vec{j}\frac{\partial}{\partial y}\left(\frac{1}{r}\right) + \vec{k}\frac{\partial}{\partial z}\left(\frac{1}{r}\right)$

$$= \vec{i}\left(\frac{-1}{r^2}\frac{\partial r}{\partial x}\right) + \vec{j}\left(\frac{-1}{r^2}\frac{\partial r}{\partial y}\right) + \vec{k}\left(\frac{-1}{r^2}\frac{\partial r}{\partial z}\right)$$

$$= \frac{-1}{r^2}\left[\vec{i}\frac{x}{r} + \vec{j}\frac{y}{r} + \vec{k}\frac{z}{r}\right]$$

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$$\nabla\left(\frac{1}{r}\right) = \frac{-1}{r^3}(\vec{r})$$

$$= \frac{-1}{r^2}\left(\frac{\vec{r}}{r}\right) = \frac{-\hat{r}}{r^2}$$

$$\therefore \boxed{\nabla\left(\frac{1}{r}\right) = \frac{-\vec{r}}{r^3} = -\frac{\hat{r}}{r^2}}$$

(iii) $$\nabla r^n = \left(\vec{i}\frac{\partial r^n}{\partial x} + \vec{j}\frac{\partial r^n}{\partial y} + \vec{k}\frac{\partial r^n}{\partial z}\right).$$

$$= \vec{i}\, n r^{n-1}\frac{\partial r}{\partial x} + \vec{j}\, n r^{n-1}\frac{\partial r}{\partial y} + \vec{k}\, n r^{n-1}\frac{\partial r}{\partial z}$$

$$= n r^{n-1}\left[\vec{i}\frac{x}{r} + \vec{j}\frac{y}{r} + \vec{k}\frac{z}{r}\right]$$

$$= \frac{n r^{n-1}}{r}\left[x\vec{i} + y\vec{j} + z\vec{k}\right]$$

$$\boxed{\nabla r^n = n r^{n-2}\,\vec{r}}$$

(iv) $$\nabla f(r) = \vec{i}\frac{\partial}{\partial x}f(r) + \vec{j}\frac{\partial}{\partial y}f(r) + \vec{k}\frac{\partial}{\partial z}f(r)$$

$$= \vec{i}\, f'(r)\frac{\partial r}{\partial x} + \vec{j}\, f'(r)\frac{\partial r}{\partial y} + \vec{k}\, f'(r)\frac{\partial r}{\partial z}$$

$$= f'(r)\left[\vec{i}\frac{x}{r} + \vec{j}\frac{y}{r} + \vec{k}\frac{z}{r}\right]$$

$$= \frac{f'(r)}{r}\left(x\vec{i} + y\vec{j} + z\vec{k}\right)$$

$$\boxed{\nabla f(r) = \frac{f'(r)}{r}\,\vec{r}}$$

(v) $$\nabla f(r) \times \vec{r} = \frac{f'(r)}{r}\,\vec{r} \times \vec{r}$$

$$= \frac{1}{r}f'(r)\left[\vec{r}\times\vec{r}\right] = 0 \quad (\because \vec{r}\times\vec{r} = 0)$$

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$$\boxed{\nabla f(r) \times \vec{r} = 0}$$

(vi) $\nabla^2 \varphi = \nabla(\nabla\varphi)$ ($\because$ $\nabla\varphi$ is solenoidal $\nabla\varphi = 0$)

$= \nabla(0)$

$$\boxed{\nabla^2 \varphi = 0}.$$

Level Surface : Important Results :

Unit Normal :

A unit normal to the given surface φ at the point is $\dfrac{\nabla\varphi}{|\nabla\varphi|}$

Directional Derivative :

The directional derivative of φ in the directio $\vec{a}$ is given by,

$$\nabla\varphi \cdot \frac{\vec{a}}{|\vec{a}|} \quad \text{(or)} \quad \nabla\varphi \cdot \hat{n} \quad \text{where } \hat{n} = \frac{\vec{a}}{|\vec{a}|}$$

The directional derivative is maximum in the direction of the normal to the given surface. Its maximum value is $|\nabla\varphi|$.

Angle between two surfaces :

$$\boxed{\cos\theta = \frac{\nabla\varphi_1 \cdot \nabla\varphi_2}{|\nabla\varphi_1|\,|\nabla\varphi_2|}}$$

Note :

If the surfaces cut orthogonally then,

$$\boxed{\nabla\varphi_1 \cdot \nabla\varphi_2 = 0}$$