

PROPERTIES:

Change of Scale property:

If $L\{f(t)\} = F(s)$, then

$$L[f(at)] = \frac{1}{a} F\left(\frac{s}{a}\right).$$

Proof:

We know that,

$$L[f(at)] = \int_0^{\infty} e^{-st} f(at) dt$$

$$\text{put } at = x \Rightarrow t = x/a$$

$$dt = dx/a$$

$$L[f(at)] = \int_0^{\infty} e^{-s(x/a)} f(x) \frac{dx}{a}$$

$$= \frac{1}{a} \int_0^{\infty} e^{-s(x/a)} f(x) dx$$

$$= \frac{1}{a} \int_0^{\infty} e^{-(s/a)x} f(x) dx$$

$$= \frac{1}{a} \int_0^{\infty} e^{-(s/a)t} f(t) dt$$

$$= \frac{1}{a} F\left(\frac{s}{a}\right)$$

First Shifting property:

If $\mathcal{L}\{f(t)\} = F(s)$ then

$$(i) \mathcal{L}[e^{-at} f(t)] = \left\{ \mathcal{L}[f(t)] \right\}_{s \rightarrow s+a} = F(s+a)$$

$$(ii) \mathcal{L}[e^{at} f(t)] = \left\{ \mathcal{L}[f(t)] \right\}_{s \rightarrow s-a} = F(s-a)$$

Proof:

(i) We know that,

$$\mathcal{L}[f(t)] = \int_0^{\infty} e^{-st} f(t) dt = F(s)$$

$$\begin{aligned} \mathcal{L}[e^{-at} f(t)] &= \int_0^{\infty} e^{-st} [e^{-at} f(t)] dt \\ &= \int_0^{\infty} e^{-(s+a)t} f(t) dt \\ &= F(s+a) \end{aligned}$$

$$\begin{aligned} (ii) \mathcal{L}[e^{at} f(t)] &= \int_0^{\infty} e^{-st} [e^{at} f(t)] dt \\ &= \int_0^{\infty} e^{-(s-a)t} f(t) dt \\ &= F(s-a) \end{aligned}$$

Second Shifting property:

If $\mathcal{L}\{f(t)\} = F(s)$ and $g(t) = \begin{cases} f(t-a), & t > a \\ 0, & t < a \end{cases}$

then $\mathcal{L}[g(t)] = e^{-as} F(s)$.

Proof:

$$\begin{aligned} \mathcal{L}[g(t)] &= \int_0^{\infty} e^{-st} g(t) dt \\ &= \int_0^a e^{-st} g(t) dt + \int_a^{\infty} e^{-st} g(t) dt \end{aligned}$$

$$L[g(t)] = 0 + \int_a^{\infty} e^{-st} f(t-a) dt$$

$$= \int_a^{\infty} e^{-st} f(t-a) dt$$

Put $t-a = u \Rightarrow dt = du$

When $t = a \Rightarrow u = 0$

$t \rightarrow \infty \Rightarrow u \rightarrow \infty$

$$L[g(t)] = \int_0^{\infty} e^{-s(u+a)} f(u) du$$

$$= \int_0^{\infty} e^{-us} e^{-as} f(u) du$$

$$= e^{-as} \int_0^{\infty} e^{-us} f(u) du$$

$$= e^{-as} \int_0^{\infty} e^{-st} f(t) dt \quad \text{Replace } u \rightarrow t$$

$$L[g(t)] = e^{-as} F(s)$$

Laplace transforms of derivatives:

If $L[f(t)] = F(s)$ then

$$L[f'(t)] = sF(s) - f(0)$$

Proof:

$$L[f'(t)] = \int_0^{\infty} e^{-st} f'(t) dt$$

Integrating by parts we get,

$$= \left[e^{-st} f(t) \right]_0^{\infty} - \int_0^{\infty} f(t) (-s e^{-st}) dt$$

$$= \lim_{t \rightarrow \infty} [e^{-st} f(t)] - e^{-s \cdot 0} f(0) + s \int_0^{\infty} e^{-st} f(t) dt$$

$$= -f(0) + s \mathcal{L}\{f(t)\}$$

$$= sF(s) - f(0)$$

Corollary:

$$\text{Let } \mathcal{L}\{f''(t)\} = s^2 F(s) - s f(0) - f'(0)$$

$$\text{Let } \mathcal{L}\{g'(t)\} = s G(s) - g(0)$$

We know that,

$$\mathcal{L}\{f'(t)\} = s \mathcal{L}\{f(t)\} - f(0)$$

Replace $f(t) \rightarrow f'(t)$ & $f'(t) \rightarrow f''(t)$ & $f(0) \rightarrow f'(0)$

$$\Rightarrow \mathcal{L}\{f''(t)\} = s \mathcal{L}\{f'(t)\} - f'(0)$$

$$= s [s \mathcal{L}\{f(t)\} - f(0)] - f'(0)$$

$$= s^2 \mathcal{L}\{f(t)\} - s f(0) - f'(0)$$

$$= s^2 F(s) - s f(0) - f'(0)$$

Laplace Transform of integrals:

$$\text{If } \mathcal{L}\{f(t)\} = F(s) \text{ then } \mathcal{L}\left[\int_0^t f(t) dt\right] = \frac{F(s)}{s}$$

Proof:

$$\text{Let } g(t) = \int_0^t f(t) dt \text{ and } g(0) = 0$$

$$\text{Then } g'(t) = f(t)$$

$$\text{WKT } \mathcal{L}\{g'(t)\} = s \mathcal{L}\{g(t)\} - g(0)$$

$$= s \mathcal{L}\{g(t)\}$$

$$\Rightarrow \mathcal{L}\{g(t)\} = \frac{1}{s} \mathcal{L}\{g'(t)\}$$

$$\Rightarrow \mathcal{L} \left[\int_0^t f(t) dt \right] = \frac{1}{s} \mathcal{L} [f(t)] \quad \left\{ \begin{array}{l} \because g(t) = \int_0^t f(t) dt \\ g'(t) = f(t) \end{array} \right.$$

$$\Rightarrow \mathcal{L} \left[\int_0^t f(t) dt \right] = \frac{F(s)}{s}$$

Derivative of Laplace Transform (or) Laplace transform of $t f(t)$:

If $\mathcal{L} [f(t)] = F(s)$ then

$$\mathcal{L} [t f(t)] = -\frac{d}{ds} F(s)$$

Proof:

We know that,

$$\mathcal{L} [f(t)] = F(s) = \int_0^{\infty} e^{-st} f(t) dt$$

$$\frac{d}{ds} F(s) = \frac{d}{ds} \int_0^{\infty} e^{-st} f(t) dt$$

$$= \int_0^{\infty} \frac{\partial}{\partial s} (e^{-st}) f(t) dt$$

$$= \int_0^{\infty} -t e^{-st} f(t) dt$$

$$= - \int_0^{\infty} e^{-st} t f(t) dt$$

$$= - \mathcal{L} [t f(t)]$$

$$\Rightarrow \mathcal{L} [t f(t)] = -\frac{d}{ds} [F(s)]$$

In general,

$$\mathcal{L} [t^n f(t)] = (-1)^n \frac{d^n}{ds^n} [F(s)]$$

Problems:

Change of Scale property:

① Find $L[\sinh 3t]$ by using change of scale property

Soln:

$$L[\sinh ht] = \frac{1}{s^2 - 1} = F(s)$$

$$L[\sinh 3t] = \frac{1}{3} F\left(\frac{s}{3}\right)$$

$$= \frac{1}{3} \frac{1}{\left(\frac{s}{3}\right)^2 - 1}$$

$$= \frac{1}{3} \left(\frac{9}{s^2 - 9}\right)$$

$$= \frac{3}{s^2 - 9}$$

② Find $L(\cos 5t)$ using change of scale property?

Soln:

$$L(\cos t) = \frac{s}{s^2 + 1} = F(s)$$

$$L(\cos 5t) = \frac{1}{5} F\left(\frac{s}{5}\right)$$

$$= \frac{1}{5} \left[\frac{s/5}{\left(\frac{s}{5}\right)^2 + 1} \right]$$

$$= \frac{1}{5} \left[\frac{5s}{s^2 + 25} \right]$$

$$= \frac{s}{s^2 + 25}$$

③ Given $L[f(t)] = \frac{s^2 - s + 1}{(2s+1)^2(s-1)}$: applying the

Change of scale property Show that

$$L[f(2t)] = \frac{s^2 - 2s + 4}{4(s+1)^2(s-2)}$$

Soln:

$$L[f(t)] = \frac{s^2 - s + 1}{(2s+1)^2(s-1)} = F(s)$$

$$L[f(2t)] = \frac{1}{2} F\left(\frac{s}{2}\right)$$

$$= \frac{1}{2} \left[\frac{\left(\frac{s}{2}\right)^2 - \left(\frac{s}{2}\right) + 1}{\left(2\left(\frac{s}{2}\right) + 1\right)^2 \left(\frac{s}{2} - 1\right)} \right]$$

$$= \frac{1}{2} \left[\frac{s^2 - 2s + 4}{4 \cdot \frac{(s+1)^2(s-2)}{2}} \right]$$

$$= \frac{1}{4} \cdot \frac{s^2 - 2s + 4}{(s+1)^2(s-2)}$$

④ Find $L[e^{5t}]$ applying change of scale property.

Soln:

$$L(e^t) = \frac{1}{s-1} = F(s)$$

$$L(e^{5t}) = \frac{1}{5} F\left(\frac{s}{5}\right)$$

$$= \frac{1}{5} \cdot \frac{1}{\frac{s}{5} - 1}$$

$$= \frac{1}{5} \cdot \frac{5}{s-5}$$

$$= \frac{1}{s-5}$$

First Shifting theorem:

① Find $L [e^{-3t} \sin^2 t]$

Proof:

$$L [e^{-at} f(t)] = F(s+a)$$

$$L [e^{-3t} \sin^2 t] = L [\sin^2 t]_{s \rightarrow s+3}$$

$$= L \left[\frac{1 - \cos 2t}{2} \right]_{s \rightarrow s+3}$$

$$= \frac{1}{2} \{ L(1) - L(\cos 2t) \}_{s \rightarrow s+3}$$

$$= \frac{1}{2} \left\{ \frac{1}{s} - \frac{s}{s^2+4} \right\}_{s \rightarrow s+3}$$

$$= \frac{1}{2} \left\{ \frac{1}{s+3} - \frac{s+3}{(s+3)^2+4} \right\}$$

$$= \frac{1}{2} \left\{ \frac{4}{(s+3)[(s+3)^2+4]} \right\}$$

$$= \frac{2}{(s+3)[(s+3)^2+4]}$$

② Find $L (t^2 e^{-2t})$.

Soln:

$$L [e^{-at} f(t)] = F(s+a)$$

$$L [e^{-2t} t^2] = [L(t^2)]_{s \rightarrow s+2}$$

$$= \left[\frac{2}{s^3} \right]_{s \rightarrow s+2}$$

$$= \frac{2}{(s+2)^3}$$

③ Find $L [e^{2t} \cos 5t]$.

Soln:

$$L [e^{2t} \cos 5t] = L [\cos 5t]_{s \rightarrow s-2}$$

$$= \left[\frac{s}{s^2 + 25} \right]_{s \rightarrow s-2}$$

$$= \frac{s-2}{(s-2)^2 + 25}$$

Second Shifting theorem:

① Find $L [f(t)]$ where $f(t) = \begin{cases} 0, & 0 < t < 2 \\ 3, & t > 2 \end{cases}$

Soln:

$$L \{f(t)\} = \int_0^{\infty} e^{-st} f(t) dt$$

$$= \int_0^2 e^{-st} f(t) dt + \int_2^{\infty} e^{-st} f(t) dt$$

$$= 0 + \int_2^{\infty} e^{-st} \cdot 3 dt$$

$$= 3 \left[\frac{e^{-st}}{-s} \right]_2^{\infty}$$

$$= \frac{-3}{s} [e^{-\infty} - e^{-2s}]$$

$$= \frac{3e^{-2s}}{s}$$

② Find the Laplace transform of

$$f(t) = \begin{cases} \sin t, & 0 < t < \pi \\ 0, & t > \pi \end{cases}$$

Soln:

$$L \{f(t)\} = \int_0^{\pi} \sin t dt + \int_{\pi}^{2\pi} 0 dt$$

$$\begin{aligned}
 \mathcal{L}[f(t)] &= \int_0^{\infty} e^{-st} f(t) dt \\
 &= \int_0^{\pi} e^{-st} f(t) dt + \int_{\pi}^{\infty} e^{-st} \cdot 0 dt \\
 &= \int_0^{\pi} \frac{e^{-st}}{s^2+1} (-s \sin t - \cos t) dt \\
 &= \frac{e^{-s\pi}}{s^2+1} (-s \sin \pi - \cos \pi) + \frac{1}{s^2+1} \\
 &= \frac{e^{-\pi s}}{s^2+1} + \frac{1}{s^2+1} = \frac{1+e^{-\pi s}}{s^2+1}
 \end{aligned}$$

Laplace Transforms of Derivatives:

(1) Find $\mathcal{L}[t \sin at]$

Soln:

$$\mathcal{L}[t \sin at] =$$

$$f(t) = t \sin at$$

$$f'(t) = at \cos at + \sin at$$

$$f''(t) = a[-at \sin at + \cos at] + a \cos at$$

$$\Rightarrow 2a \cos at - a^2 t \sin at$$

$$f(0) = 0, f'(0) = 0$$

$$\mathcal{L}[f''(t)] = s^2 \mathcal{L}[f(t)] - sf(0) - f'(0)$$

$$\mathcal{L}[2a \cos at - a^2 t \sin at] = s^2 \mathcal{L}[t \sin at] - s(0) - 0$$

$$\Rightarrow 2a \mathcal{L}[\cos at] - a^2 \mathcal{L}[t \sin at] = s^2 \mathcal{L}[t \sin at]$$

$$\Rightarrow (s^2 + a^2) \mathcal{L}[t \sin at] = 2a \mathcal{L}[\cos at]$$

$$\Rightarrow (s^2 + a^2) \mathcal{L}[t \sin at] = 2a \cdot \frac{s}{a^2 + s^2}$$

$$\mathcal{L}[t \sin at] = \frac{2as}{(s^2 + a^2)^2}$$

① Find $L [t \cos at]$

Soln:

$$L [t f(t)] = -\frac{d}{ds} [L(f(t))]$$

$$L [t \cos at] = -\frac{d}{ds} [L(\cos at)]$$

$$= -\frac{d}{ds} \left[\frac{s}{s^2 + a^2} \right]$$

$$= - \left\{ \frac{s^2 + a^2 - s(2s)}{(s^2 + a^2)^2} \right\}$$

$$= - \left[\frac{a^2 - s^2}{(s^2 + a^2)^2} \right]$$

$$= \frac{s^2 - a^2}{(s^2 + a^2)^2}$$

② Find $L [te^{2t} \sin 3t]$

Soln:

$$L [te^{2t} \sin 3t] = -\frac{d}{ds} \{ L [e^{2t} \sin 3t] \}$$

$$= -\frac{d}{ds} \{ L(\sin 3t) \}_{s \rightarrow s-2}$$

$$= -\frac{d}{ds} \left\{ \left(\frac{3}{s^2 + 9} \right)_{s \rightarrow s-2} \right\}$$

$$= - \left\{ \frac{0 - 3(2s)}{(s^2 + 9)^2} \right\}_{s \rightarrow s-2}$$

$$= \left\{ \frac{6s}{(s^2 + 9)^2} \right\}_{s \rightarrow s-2}$$

$$= \frac{6(s-2)}{[(s-2)^2+9]^2}$$

$$= \frac{6(s-2)}{(s^2-4s+13)^2}$$

③ Find $L[t^2 e^{-2t} \cos t]$

Soln:

$$L[t^2 e^{-2t} \cos t] = (-1)^2 \frac{d^2}{ds^2} \{ L(e^{-2t} \cos t) \}$$

$$= \frac{d^2}{ds^2} \left\{ L(\cos t)_{s \rightarrow s+2} \right\}$$

$$= \frac{d^2}{ds^2} \left\{ \frac{s}{s^2+1} \right\}_{s \rightarrow s+2}$$

$$= \frac{d}{ds} \left\{ \frac{s^2+1-2s^2}{(s^2+1)^2} \right\}_{s \rightarrow s+2}$$

$$= \frac{d}{ds} \left\{ \frac{1-s^2}{(s^2+1)^2} \right\}_{s \rightarrow s+2}$$

$$= \left\{ \frac{(s^2+1)^2(-2s) - (1-s^2)2(s^2+1)(2s)}{(s^2+1)^4} \right\}_{s \rightarrow s+2}$$

$$= \left\{ \frac{(s^2+1)(-2s) - 4s(1-s^2)}{(s^2+1)^3} \right\}_{s \rightarrow s+2}$$

$$= \frac{[(s+2)^2+1] [-2(s+2)] - 4(s+2)[1-(s+2)^2]}{[(s+2)^2+1]^3}$$

$$= \frac{(s^2+4s+5) [-2s-4] + (4s+8)(s^2+4s+3)}{[(s+2)^2+1]^3}$$

$$= \frac{2s^3 + 12s^2 + 18s + 4}{(s^2+4s+5)^3}$$

$$(s^2+4s+5)^3$$

(4) Find $L \left[\frac{\sin 3t}{t} \right]$

Soln:

$$L \left[\frac{f(t)}{t} \right] = \int_s^\infty F(s) ds = \int_s^\infty L[f(t)] ds$$

$$L \left[\frac{\sin 3t}{t} \right] = \int_s^\infty L(\sin 3t) ds$$

$$= \int_s^\infty \left(\frac{3}{s^2 + 9} \right) ds$$

$$= \int_s^\infty \frac{3}{s^2 + 3^2} ds$$

$$= 3 \cdot \frac{1}{3} \left[\tan^{-1} \left(\frac{s}{3} \right) \right]_s^\infty$$

$$\left(\because \int \frac{dx}{x^2 + a^2} \right)$$

$$= \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right)$$

$$= \tan^{-1}(\infty) - \tan^{-1}(s/3)$$

$$= \pi/2 - \tan^{-1}(s/3)$$

$$= \cot^{-1}(s/3)$$