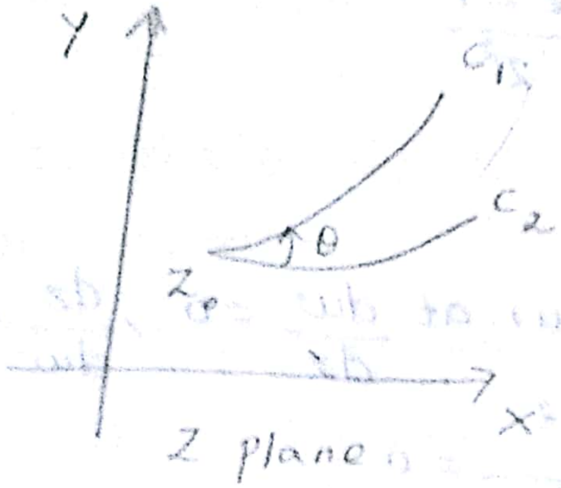
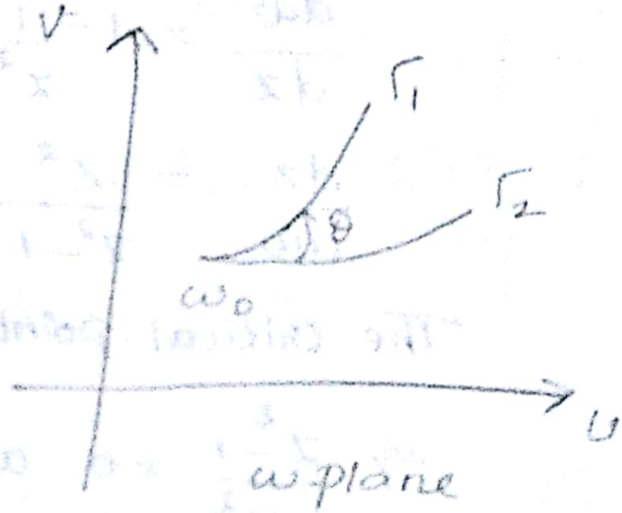
Conformal mapping :

Defn:

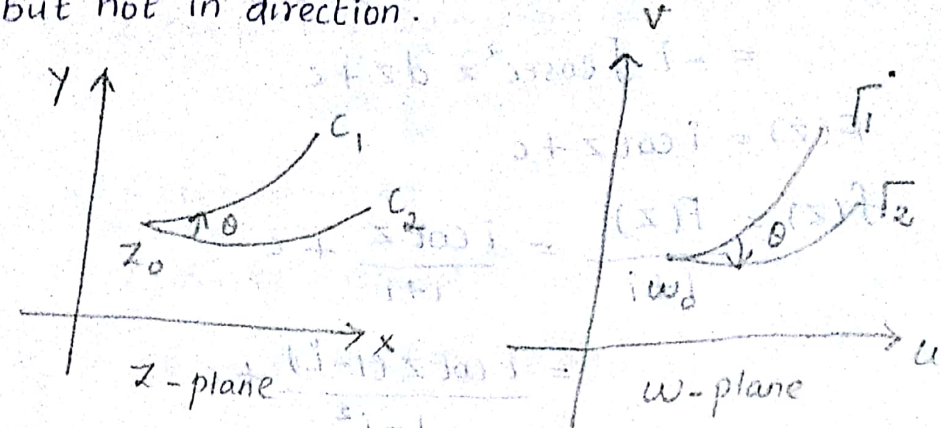
A mapping $w = f(z)$ is said to be Conformal at $z = z_0$ if it preserves the angle between any two curves through z_0 in z plane both in magnitude and direction.





Isogonal mapping :

A mapping $w = f(z)$ is said to be isogonal at $z = z_0$ if it preserves the angle between any two curves through z_0 in z plane only in magnitude but not in direction.



Remarks :

1. If $f(z)$ is analytic and $f'(z) \neq 0$ at each Point then the mapping $w = f(z)$ is conformal.
2. The points at which $w = f(z)$ is not conformal i.e., $f'(z) = 0$ are called Critical points.
3. The critical points of $w = f(z)$ will occur at $\frac{dz}{dw} = 0$ and $\frac{dw}{dz} = 0$.

① Find the critical points of the transformation $w = z + 1/z$.

Soln:

$$\frac{dw}{dz} = 1 - \frac{1}{z^2} = \frac{z^2 - 1}{z^2}$$

$$\frac{dz}{dw} = \frac{z^2}{z^2 - 1}$$

The critical points occur at $\frac{dw}{dz} = 0$, $\frac{dz}{dw} = 0$.

$$\Rightarrow \frac{z^2 - 1}{z^2} = 0 \text{ and } \frac{z^2}{z^2 - 1} = 0$$

$$\Rightarrow z^2 - 1 = 0 \text{ and } z^2 = 0$$

$$\Rightarrow z = \pm 1 \text{ and } z = 0$$

$\therefore z = 0, 1, -1$ are the critical points.

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(2) Find the critical points of $w^2 = (z-\alpha)(z-\beta)$.

Soln:

$$w^2 = (z-\alpha)(z-\beta)$$

$$2w\frac{dw}{dz} = (z-\alpha) + (z-\beta)$$

$$\frac{dw}{dz} = \frac{(z-\alpha)+(z-\beta)}{2w}$$

$$\frac{dz}{dw} = \frac{2w}{(z-\alpha)+(z-\beta)}$$

$$\frac{dw}{dz} = \frac{(z-\alpha)+(z-\beta)}{2w} \quad \& \quad \frac{dz}{dw} = \frac{2w}{(z-\alpha)+(z-\beta)}$$

$\therefore$ The critical points occur at $\frac{dw}{dz} = 0$ and $\frac{dz}{dw} = 0$.

$$\Rightarrow \frac{z-\alpha+z-\beta}{2w} = 0 \quad \text{and} \quad \frac{2w}{(z-\alpha)+(z-\beta)} = 0$$

$$\Rightarrow 2z = \alpha+\beta \qquad w = 0$$

$$\Rightarrow z = \frac{\alpha+\beta}{2} \qquad \Rightarrow w^2 = 0$$

$$\Rightarrow (z-\alpha)(z-\beta) = 0$$

$$\Rightarrow z = \alpha, \beta$$

$\therefore z = \alpha, \beta, \frac{\alpha+\beta}{2}$ are the critical points.

(3) Find the points such that $w = f(z) = \sin z$ is not conformal.

Soln: Let $w = \sin z$

$$\frac{dw}{dz} = \cos z \; ; \quad \frac{dz}{dw} = \frac{1}{\cos z}$$

The critical points occur at $\frac{dw}{dz} = 0$ and $\frac{dz}{dw} = 0$.

$$\Rightarrow \cos z = 0 \quad \text{and} \quad 1/\cos z = 0$$

$$\Rightarrow z = \cos^{-1}(0) \qquad 1 = 0, \text{ impossible}$$

$$z = \pm(2n-1)\frac{\pi}{2}, \quad n = 0, 1, 2, \ldots$$

Conformal mapping :

1) Find the image of the following region under the translation $w = 1/z$

(i) half plane $x > c$ when $c > 0$

(ii) the infinite strip $\frac{1}{4} < y < 1/2$

(iii) the infinite strip $0 < y < 1/2$.

Soln: $w = \frac{1}{z}$

$$z = \frac{1}{w}$$

$$x + iy = \frac{1}{u+iv} = \frac{1}{u+iv} \cdot \frac{u-iv}{u-iv}$$

$$x + iy = \frac{u-iv}{u^2+v^2} = \frac{u}{u^2+v^2} - i\frac{v}{u^2+v^2}$$

$$x = \frac{u}{u^2+v^2}, \quad y = \frac{-v}{u^2+v^2}$$

(i) Half plane $x > c$ when $c > 0$

$$x = c$$

$$\frac{u}{u^2+v^2} = c$$

$$u = c(u^2+v^2)$$

$$\frac{u}{c} = u^2 + v^2$$

$$u^2 - \frac{u}{c} + v^2 = 0.$$

$$\left(u^2 - \frac{u}{c} + \left(\frac{1}{2c}\right)^2\right) + v^2 - \left(\frac{1}{2c}\right)^2 = 0$$

$$\left(u - \frac{1}{2c}\right)^2 + v^2 = \left(\frac{1}{2c}\right)^2$$

which is a circle with centre $\left(\frac{1}{2c}, 0\right)$ & radius $1/2c$.

$u^2 - \frac{u}{c}$

$a = u$

$2ab = \frac{u}{c}$

$b = \frac{u}{2ac}$

$b = \frac{\cancel{u}}{2\cancel{u}c}$

$b = \frac{1}{2c}$

(ii) the infinite strip $\frac{1}{4} < y < \frac{1}{2}$

$y = \frac{1}{4}$

$\frac{-v}{u^2+v^2} = \frac{1}{4}$

$-v = \frac{1}{4}(u^2+v^2)$

$-4v = u^2+v^2$

$u^2+v^2+4v = 0$

$u^2+(v+2)^2-4=0$

$u^2+(v+2)^2 = 4$

which is a eqn of circle with centre $(0,-2)$ & $r = 2$

$y = \frac{1}{2}$

$\frac{-v}{u^2+v^2} = \frac{1}{2}$

$-2v = u^2+v^2$

$u^2+v^2+2v = 0$

$u^2+(v+1)^2-1=0$

$u^2+(v+1)^2 = 1$

centre: $(0,-1)$

$r = 1$

(iii) $0 < y < 1/2$

$y = 0$

$\frac{-v}{u^2+v^2} = 0$

$\boxed{v = 0}$

which is a straight line in w-plane.

$y = \frac{1}{2}$

$\frac{-v}{u^2+v^2} = \frac{1}{2}$

$-2v = u^2+v^2$

$u^2+v^2+2v=0$

$u^2+(v+1)^2-1=0$

$u^2+(v+1)^2 = 1$

centre : $(0,-1)$

radius = 1