Numerical Integration by Trapezoidal Rule :

Trapezoidal Rule :

$$\int_{x_0}^{x_n} y\, dx = \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + \cdots + y_{n-1})\right]$$

where $h = \frac{b-a}{n}$

1]. Using trapezoidal rule, evaluate $\int_{-1}^{1} \frac{dx}{1+x^2}$ taking 8 intervals.

Soln.

Given $y = \frac{1}{1+x^2}$ and $h = \frac{b-a}{n}$ $\quad a=-1$, $b=1$

$h = \frac{1+1}{8} = 0.25$

x :	-1	-0.75	-0.5	-0.25	0	0.25	0.5	0.75	1
y :	0.5	0.64	0.8	0.9412	1	0.9412	0.8	0.64	0.5

By Trapezoidal rule,

$$\int_{-1}^{1} \frac{1}{1+x^2}\, dx = \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + \cdots + y_{n-1})\right]$$

$$= \frac{0.25}{2}\left[(0.5 + 0.5) + 2(0.64 + 0.8 + 0.9412 + 1 + 0.9412 + 0.8 + 0.64)\right]$$

$$= \frac{0.25}{2} \times 12.5248$$

$$= 1.5656$$

2]. Dividing the range into 10 equal parts, find the value of $\int_{0}^{\pi/2} \sin x\, dx$ by Trapezoidal rule.

Soln.

Given $y = \sin x$ and $h = \frac{b-a}{n} = \frac{\pi/2 - 0}{10} = \frac{\pi}{20}$

x :	0	$\pi/20$	$2\pi/20$	$3\pi/20$	$4\pi/20$	$5\pi/20$	$6\pi/20$	$7\pi/20$	$8\pi/20$	$\frac{9\pi}{20}$	$\frac{10\pi}{20}$
y :	0	0.1564	0.3090	0.4540	0.5878	0.7071	0.8090	0.8910	0.9511	0.9877	1

By Trapezoidal rule,

$$\int_0^{\pi/2} \sin x \, dx = \frac{h}{2}\left[(y_0+y_n)+2(y_1+y_2+\dots+y_{n-1})\right]$$

$$= \frac{\pi/20}{2}\left[(0+1)+2(0.1564+0.3090+0.4540+0.5878+0.7071+0.8090+0.8910+0.9511+0.9877)\right]$$

$$= \frac{\pi}{40}[12.7062]$$

$$= 0.9980$$

7. Evaluate $\int_0^1 \frac{dx}{1+x^2}$, using Trapezoidal rule with $h=0.2$. Hence obtain an approximate value of π.

Soln.

Given $y = \frac{1}{1+x^2}$ and $h = 0.2$

x :	0	0.2	0.4	0.6	0.8	1
y :	1	0.9615	0.8621	0.7353	0.6098	0.5

By Trapezoidal rule,

$$\int_0^1 \frac{dx}{1+x^2} = \frac{h}{2}\left[(y_0+y_n)+2(y_1+y_2+\dots+y_{n-1})\right]$$

$$= \frac{0.2}{2}\left[(1+0.5)+2(0.9615+0.8621+0.7353+0.6098)\right]$$

$$= 0.7837$$

To find the value of π :

By Actual Integration,

$$\int_0^1 \frac{dx}{1+x^2} = \left[\tan^{-1} x\right]_0^1 = \tan^{-1}(1) - \tan^{-1}(0)$$

$$= \frac{\pi}{4} - 0$$

$$= \pi/4$$

$$= 0.7837$$

$[\pi = 3.1348$ approximately$]$

Simpson's $1/3^{rd}$ rule:

$$\int_{x_0}^{x_n} y\,dx = \frac{h}{3}\left[(y_0+y_n)+2(y_2+y_4+\cdots)+4(y_1+y_3+\cdots)\right]$$

$$= \frac{h}{3}\left\{\begin{array}{l}[\text{Sum of the } 1^{st} \text{ and last ordinates}] \\ +2[\text{Sum of the remaining odd ordinates}] \\ +4[\text{Sum of the even ordinates}]\end{array}\right\}$$

Note:

1. This formula can be applied only if the number of intervals is even (or) the number of ordinates is odd.

Simpson's $3/8^{th}$ rule:

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{3h}{8}\left[(y_0+y_n)+3(y_1+y_2+y_4+y_5+\cdots+y_{n-1}) + 2(y_3+y_6+y_9+\cdots+y_n)\right]$$

which is applicable only when n is a multiple of 3.

1. Dividing the range into 10 equal parts, find the value of $\int_0^{\pi/2} \sin x\,dx$ by Simpson's $1/3^{rd}$ rule.

Soln.

Given $y=\sin x$ and $h = \frac{b-a}{n} = \frac{\pi/2-0}{10} = \pi/20$

x:	0	$\pi/20$	$2\pi/20$	$3\pi/20$	$4\pi/20$	$5\pi/20$	$6\pi/20$	$7\pi/20$	$8\pi/20$	$9\pi/20$	$10\pi/20$
y:	0	0.1564	0.3090	0.4540	0.5878	0.7071	0.8090	0.8910	0.9511	0.9877	1

By Simpson's $1/3^{rd}$ rule,

$$\int_0^{\pi/2} \sin x\,dx = \frac{h}{3}\left[(y_0+y_{10})+4(y_1+y_3+y_5+y_7+y_9) + 2(y_2+y_4+y_6+y_8)\right]$$

$$= \frac{\pi/20}{3}\left[(0+1)+4(0.1564+0.4540+0.7071+0.8910+0.9877) + 2(0.3090+0.5878+0.809+0.9511)\right]$$

$$= \frac{\pi}{60}[19.0986]$$

$$= 1.0000$$

2]. Evaluate $I = \int_0^6 \frac{1}{1+x}\, dx$ using i). Trapezoidal rule ii). Simpson's rule. Also, check by direct integration

Soln.

Given $y = \frac{1}{1+x}$ and $h = \frac{b-a}{n} = \frac{6-0}{6} = 1$

Take the number of intervals n as 6.

x :	0	1	2	3	4	5	6
y :	1	0.5	0.3333	0.25	0.2	0.1667	0.1429

i). By Trapezoidal rule,

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{2}\{(y_0+y_n) + 2(y_1+y_2+\cdots+y_{n-1})\}$$

$$= \frac{1}{2}\{(1+0.1429) + 2(0.5+0.3333+0.25+0.2+0.1667)\}$$

$$= \frac{1}{2}\{4.0429\}$$

$$= 2.0215$$

ii). By Simpson's $1/3^{rd}$ rule

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{3}\{(y_0+y_6) + 2(y_2+y_4+\cdots) + 4(y_1+y_3+y_5+\cdots)]$$

$$= \frac{1}{3}\{(1+0.1429) + 2(0.3333+0.2) + 4(0.5+0.25+0.1667)]$$

$$= \frac{1}{3}(5.8763)$$

$$= 1.9588$$

iii). By Simpson's $3/8^{th}$ rule,

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{3h}{8}\{(y_0+y_n) + 3(y_1+y_2+y_4+y_5+\cdots) + 2(y_3+y_6+y_9+\cdots)\}$$

$$= \frac{3}{8}\{(1+0.1429) + 3(0.5+0.3333+0.2+0.1667) + 2(0.25)\}$$

$$= \frac{3}{8}(4.3429) = 1.9661$$

iv). By actual Integration,

$$\int_0^6 \frac{dx}{1+x} = \left[\log(1+x)\right]_0^6 = \log 7 - \log 1$$

$$= 1.9459 - 0 \qquad (\because \ln 7)$$

$$= 1.9459$$

5. Compute $\int_4^{5.2} \log_e x \, dx$, using Simpson's 1/3 and 3/8 rule.

Soln.

Given $y = \log_e x$ and $h = \frac{b-a}{n} = \frac{5.2-4}{6} = \frac{1.2}{6}$

$h = 0.2$

x:	4	4.2	4.4	4.6	4.8	5.0	5.2
y:	1.386	1.435	1.482	1.526	1.569	1.609	1.649

By Simpson's 1/3rd rule,

$$I = \frac{h}{3}\left[(y_0 + y_n) + 2(y_2 + y_4 + \cdots) + 4(y_1 + y_3 + \cdots)\right]$$

$$= \frac{0.2}{3}\left[(1.386 + 1.649) + 2(1.482 + 1.569) + 4(1.435 + 1.526 + 1.609)\right]$$

$$= \frac{0.2}{3}\left[27.417\right]$$

$I = 1.828$

By Simpson's 3/8th rule,

$$I = \frac{3h}{8}\left[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + \cdots) + 2(y_3 + y_6 + y_9 + \cdots)\right]$$

$$= \frac{3(0.2)}{8}\left[(1.386 + 1.649) + 3(1.435 + 1.482 + 1.569 + 1.609) + 2(1.526)\right]$$

$$= 1.828$$

4]. Find the value of $\log_e 5$ from $\int_0^5 \frac{dx}{4x+5}$ by Simpson's $1/3^{rd}$ rule $(n=10)$

Soln.

Given $y = \frac{1}{4x+5}$; $h = \frac{b-a}{n} = \frac{5-0}{10} = 0.5$

x :	0	0.5	1	1.5	2	2.5	3	3.5	4	4.5	5
y :	0.2	0.1429	0.1111	0.0909	0.0769	0.0667	0.0588	0.0526	0.047	0.0434	0.04

By Simpson's $1/3^{rd}$ rule,

$$\int_0^5 \frac{dx}{4x+5} = \frac{h}{3}\left[(y_0+y_n) + 2(y_2+y_4+y_6+\dots) + 4(y_1+y_3+y_5+\dots)\right]$$

$$= \frac{0.5}{3}\left[(0.2+0.04) + 2(0.1111+0.0769+0.0588+0.047) + 4(0.1429+0.0909+0.0667+0.0526+0.0434)\right]$$

$$= \frac{0.5}{3}(2.414)$$

$$= 0.402 \rightarrow (1)$$

Now,

$$\int_0^5 \frac{dx}{4x+5} = \left[\frac{\log(4x+5)}{4}\right]_0^5$$

$$= \frac{1}{4}(\log 25 - \log 5)$$

$$= \frac{1}{4}\log\left(\frac{25}{5}\right) = \frac{1}{4}\log(5) \rightarrow (2)$$

From (1) & (2), $\frac{1}{4}\log 5 = 0.402$ $(\because \ln 5)$

$\log 5 = 1.608$