Numerical methods

Unit - III

Numerical Differentiation and Numerical Integration

* Approximation of derivatives using interpolation polynomials
* Differentiation using interpolation formulae
* Numerical integration by Trapezoidal and Simpson's $1/3^{rd}$ and $3/8^{th}$ rules
* Double integrals using Trapezoidal and Simpson's rules.

Numerical Differentiation:

It is the process of computing the value of the derivative $\frac{dy}{dx}$ for some particular value of x, from the given data (x_i, y_i).

If the values of x are equally spaced, we can use Newton's interpolation formula for equal intervals.

If the values of x are unequally spaced, we can use Lagrange's interpolation formula (or) Newton's divided difference interpolation formula.

Differentiation using interpolation formulae:

Newton's forward difference formula to compute the derivatives:

Let us consider Newton's forward difference formula

$$y = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots$$

where $u = \frac{x - x_0}{h}$

ie, $$y = y_0 + u\Delta y_0 + \frac{(u^2-u)}{2}\Delta^2 y_0 + \frac{(u^3-3u^2+2u)}{6}\Delta^3 y_0 + \cdots$$

Now,

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$$

$$= \frac{1}{h}\left[\Delta y_0 + \frac{2u-1}{2}\Delta^2 y_0 + \frac{(3u^2-6u+2)}{6}\Delta^3 y_0 + \cdots\right] \to (1)$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\Delta^2 y_0 + (u-1)\Delta^3 y_0 + \frac{(6u^2-18u+11)}{12}\Delta^4 y_0 + \cdots\right] \to (2)$$

$$\frac{d^3y}{dx^3} = \frac{1}{h^3}\left[\Delta^3 y_0 + \frac{12u-18}{12}\Delta^4 y_0 + \cdots\right] \to (3)$$

In particular, at $x = x_0$, $u = 0$. Hence putting $u=0$ in (1), (2) & (3), we get the values of 1st, 2nd & 3rd derivatives at $x = x_0$.

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \left(\frac{dy}{dx}\right)_{u=0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{\Delta^3 y_0}{3} - \cdots\right]$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_0} = \left(\frac{d^2y}{dx^2}\right)_{u=0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \cdots\right]$$

$$\left(\frac{d^3y}{dx^3}\right)_{x=x_0} = \left(\frac{d^3y}{dx^3}\right)_{u=0} = \frac{1}{h^3}\left[\Delta^3 y_0 - \frac{3}{2}\Delta^4 y_0 + \cdots\right]$$

Newton's Backward difference formula to compute the derivatives:

$$\frac{dy}{dx} = \frac{1}{h}\left\{\nabla y_n + \frac{2v+1}{2}\nabla^2 y_n + \frac{(3v^2+6v+2)}{6}\nabla^3 y_n + \frac{4v^3+18v^2+22v+6}{24}\nabla^4 y_n + \cdots\right\}$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left\{\nabla^2 y_n + (v+1)\nabla^3 y_n + \frac{6v^2+18v+11}{12}\nabla^4 y_n + \cdots\right\}$$

$$\frac{d^3y}{dx^3} = \frac{1}{h^3}\left\{\nabla^3 y_n + \frac{12v+18}{12}\nabla^4 y_n + \cdots\right\}$$

In particular, at $x = x_n$, $v = 0$. Then

$$\left(\frac{dy}{dx}\right)_{x=x_n} = \frac{1}{h}\left\{\nabla y_n + \frac{\nabla^2 y_n}{2} + \frac{\nabla^3 y_n}{3} + \frac{\nabla^4 y_n}{4} + \cdots\right\}$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_n} = \frac{1}{h^2}\left\{\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \cdots\right\}$$

$$\left(\frac{d^3y}{dx^3}\right)_{x=x_n} = \frac{1}{h^3}\left\{\nabla^3 y_n + \frac{3}{2}\nabla^4 y_n + \cdots\right\}$$

1]. Compute $f''(0)$ and $f'(0.2)$ from the following tabular data.

x :	0.0	0.2	0.4	0.6	0.8	1.0
$f(x)$:	1.00	1.16	3.56	13.96	41.96	101.00

Soln.

Since $x=0$ and 0.2 both are nearer to the initial value of the table, So we use newton's forward differences.

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$	$\Delta^5 f(x)$
0.0	1.00					
		0.16				
0.2	1.16		2.24			
		2.40		5.76		
0.4	3.56		8.00		3.84	
		10.40		9.60		0
0.6	13.96		17.60		3.84	
		28.00		13.44		
0.8	41.96		31.04			
		59.04				
1.0	101.00					

Here $u = \dfrac{x-x_0}{h} = \dfrac{0-0}{0.2} = 0$

$$\therefore \ f''(x) = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0\right]$$

$$f''(0) = \frac{1}{(0.2)^2}\left[2.24 - 5.76 + \frac{11}{12}(3.84) - \frac{5}{6}(0)\right]$$

$$= 0$$

and

$$f'(x) = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0\right]$$

$$= \frac{1}{0.2}\left(2.40 - \frac{8.00}{2} + \frac{9.60}{3} - \frac{3.84}{4}\right)$$

$$= 3.2$$

2]. Find $y'(2.2)$ & $y''(2.2)$ from the table.

x :	1.4	1.6	1.8	2.0	2.2
$y(x)$:	4.0552	4.9530	6.0496	7.3891	9.0250

Soln.

Since $x = 2.2$ which is nearer to the end value of the table, So we use newton's backward difference formula.

x	$y(x)$	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
1.4	4.0552				
		0.8978			
1.6	4.9530		0.1988		
		1.0966		0.0441	
1.8	6.0496		0.2429		0.0094
		1.3395		0.0535	
2.0	7.3891		0.2964		
		1.6359			
2.2	9.0250				

Here $v = \frac{x - x_n}{h} = \frac{2.2 - 2.2}{0.2} = 0$

$$\therefore \; y'(x) = \frac{1}{h}\left(\nabla y_n + \frac{\nabla^2 y_n}{2} + \frac{\nabla^3 y_n}{3} + \frac{\nabla^4 y_n}{4}\right)$$

$$y'(2.2) = \frac{1}{0.2}\left[1.6359 + \frac{0.2964}{2} + \frac{0.0535}{3} + \frac{0.0094}{4}\right]$$

$$= 5\,(1.8043)$$

$$= 9.0215$$

and $$y''(2.2) = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n\right]$$

$$y''(2.2) = \frac{1}{(0.2)^2}\left[0.2964 + 0.0535 + \frac{11}{12}(0.0094)\right]$$

$$= 25\,(0.3585)$$

$$= 8.9629.$$

(5). The population of a certain town is given below. Find the rate of growth of the population in 1931, 1941, 1961 and 1971.

Year (x) :	1931	1941	1951	1961	1971
Population in thousands (y) :	40.62	60.80	79.95	103.56	132.65

Soln.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1931	40.62				
		20.18			
1941	60.80		-1.03		
		19.15		5.49	
1951	79.95		4.46		-4.47
		23.61		1.02	
1961	103.56		5.48		
		29.09			
1971	132.65				

i). To find $f'(1931)$ and $f'(1941)$, we use forward formula.

Here $u = \frac{x-x_0}{h} = \frac{1931-1931}{10} = 0$

$$\left(\frac{dy}{dx}\right)_{u=0} = \frac{1}{h}\left[\Delta y_0 - \frac{\Delta^2 y_0}{2} + \frac{\Delta^3 y_0}{3} - \frac{\Delta^4 y_0}{4} + \cdots\right]$$

$$= \frac{1}{10}\left\{20.18 - \frac{(-1.03)}{2} + \frac{5.49}{3} - \frac{(-4.47)}{4}\right]$$

$$= \frac{1}{10}\left\{20.18 + 0.515 + 1.83 + 1.1175\right\}$$

$$= 2.3643$$

ii). Here $u = \frac{x-x_0}{h} = \frac{1941-1931}{10} = +1$

$$\left(\frac{dy}{dx}\right) = \frac{1}{h}\left\{\Delta y_0 + \frac{2u-1}{2}\Delta^2 y_0 + \frac{3u^2-6u+2}{6}\Delta^3 y_0 + \frac{4u^3-18u^2+22u-6}{24}\Delta^4 y_0 + \cdots\right\}$$

$$= \frac{1}{10}\left\{20.18 + \frac{1}{2}(-1.03) - \frac{1}{6}(5.49) + \frac{1}{12}(-4.47)\right\}$$

$$= \frac{1}{10}\{20.18 - 0.515 - 0.915 - 0.3725\}$$

$$= 1.83775$$

iii). To find $y'(1961)$ and $y'(1971)$, we use newton's backward formula.

$$v = \frac{x - x_n}{h} = \frac{1961-1971}{10} = -1$$

$$\frac{dy}{dx} = \frac{1}{h}\left[\nabla y_n + \frac{2v+1}{2}\nabla^2 y_n + \frac{3v^2+6v+2}{6}\nabla^3 y_n + \frac{4v^3+18v^2+22v+6}{24}\nabla^4 y_n + \cdots\right]$$

$$= \frac{1}{10}\left[29.09 - \frac{1}{2}(5.48) - \frac{1}{6}(1.02) - \frac{1}{2}(-4.47)\right]$$

$$= \frac{1}{10}[29.09 - 2.74 - 0.17 + 2.235] = \frac{28.415}{10}$$

$$= 2.8415$$

iv). To find $y'(1971)$

$$v = \frac{x - x_n}{h} = \frac{1971-1971}{10} = 0$$

$$\left(\frac{dy}{dx}\right)_{v=0} = \frac{1}{h}\left\{\nabla y_n + \frac{\nabla^2 y_n}{2} + \frac{\nabla^3 y_n}{3} + \frac{\nabla^4 y_n}{4} + \cdots\right]$$

$$= \frac{1}{10}\left[29.09 + \frac{5.48}{2} + \frac{1.02}{3} + \frac{(-4.47)}{4}\right]$$

$$= \frac{1}{10}[29.09 + 2.74 + 0.34 - 1.1175]$$

$$= \frac{31.0525}{10}$$

$$= 3.10525$$

4. A jet fighter's position on an aircraft carrier's runway was timed during landing.

t(sec) : 1.0 1.1 1.2 1.3 1.4 1.5 1.6

y(m) : 7.989 8.403 8.781 9.129 9.451 9.750 10.031

where y is the distance from the end of the carrier. Estimate velocity $\left(\frac{dy}{dt}\right)$ and acceleration $\left(\frac{d^2y}{dt^2}\right)$ at i). t=1.1 ii). t=1.6 using numerical differentiation.

3.9483 2.751

-3.5833 -1.1167

Soln,

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
1.0	7.989						
		0.414					
1.1	8.403		-0.036				
		0.378		0.006			
1.2	8.781		-0.03		-0.002		
		0.348		0.004		0.001	
1.3	9.129		-0.026		-0.001		0.002
		0.322		0.003		0.003	
1.4	9.451		-0.023		0.002		
		0.299		0.005			
1.5	9.750		-0.018				
		0.281					
1.6	10.031						

i). To find t=1.1

Here $h=0.1$

$$u = \frac{x-x_0}{h} = \frac{1.1-1.0}{0.1} = 1$$

$$\left(\frac{dy}{dx}\right)_{t=1.1} = \left(\frac{dy}{dx}\right)_{u=1} = \frac{1}{h}\Big[\Delta y_0 + \left(\frac{2u-1}{2}\right)\Delta^2 y_0 +$$

$$\left(\frac{3u^2-6u+2}{6}\right)\Delta^3 y_0 + \left(\frac{4u^3-18u^2+22u-6}{24}\right)\Delta^4 y_0 +$$

$$+ \frac{5u^4-40u^3+105u^2-100u+24}{120}\Delta^5 y_0$$

$$+ \frac{6u^5-75u^4+340u^3-675u^2+548u-120}{720}\Delta^6 y_0\Big]$$

$$= \frac{1}{0.1}\left[0.414 + \frac{1}{2}(-0.036) - \frac{1}{6}(0.006) + \frac{1}{6}(-0.002) - \frac{1}{20}(0.001) + \frac{1}{30}(0.002)\right]$$

$$= \frac{1}{0.1}(0.39468)$$

$$= 3.9468$$

ii). $$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\Delta^2 y_0 + (u-1)\Delta^3 y_0 + \left(\frac{6u^2-18u+11}{12}\right)\Delta^4 y_0 + \left(\frac{2u^3-12u^2+21u-10}{12}\right)\Delta^5 y_0 + \left(\frac{30u^4-300u^3+1020u^2-1350u+548}{720}\right)\Delta^6 y_0\right]$$

$$= \frac{1}{(0.1)^2}\left[-0.036 + 0 + \frac{1}{12}(-0.002) + \frac{1}{12}(0.001) - \frac{13}{180}(0.002)\right]$$

$$= \frac{1}{(0.1)^2}(-0.03589)$$

$$= -3.5894$$

iii). To find $t = 1.6$

Here $h = 0.1$

$$v = \frac{x - x_n}{h} = \frac{1.6 - 1.6}{0.1} = 0$$

$$\left(\frac{dy}{dx}\right)_{t=1.6} = \left(\frac{dy}{dx}\right)_{v=0} = \frac{1}{h}\left\{\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \frac{1}{5}\nabla^5 y_n + \frac{1}{6}\nabla^6 y_n\right\}$$

$$= \frac{1}{0.1}\left\{0.281 - \frac{0.018}{2} + \frac{0.005}{3} + \frac{0.002}{4} + \frac{0.003}{5} + \frac{0.002}{6}\right\}$$

$$= \frac{1}{0.1}(0.2751)$$

$$= 2.7510$$

and $\left(\frac{d^2y}{dx^2}\right)_{t=1.6} = \left(\frac{d^2y}{dx^2}\right)_{v=0}$

$$= \frac{1}{h^2}\left\{\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \frac{5}{6}\nabla^5 y_n + \frac{137}{180}\nabla^6 y_n + \dots\right\}$$

$$= \frac{1}{(0.1)^2}\left\{-0.018 + 0.005 + \frac{11}{12}(0.002) + \frac{5}{6}(0.003) + \frac{137}{180}(0.002)\right\}$$

$$= \frac{1}{(0.1)^2}(-0.0071)$$

$$= -0.7144$$

5]. Using the following data, find $f'(5)$ & $f''(5)$ and the maximum value of $f(x)$.

x :	0	2	3	4	7	9
$f(x)$:	4	26	58	112	466	922

Soln.:

Since the values of x are not equally spaced, we use newton's divided difference formula.

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$	$\Delta^5 f(x)$
0	4					
		$\frac{26-4}{2}=11$				
2	26		$\frac{32-11}{3-0}=7$			
		$\frac{58-26}{3-2}=32$		$\frac{11-7}{4-0}=1$		
3	58		$\frac{54-32}{4-2}=11$		0	
		$\frac{112-58}{4-3}=54$		$\frac{16-11}{7-2}=1$		0
4	112		$\frac{118-54}{7-3}=16$		0	
		$\frac{466-112}{7-4}=118$		$\frac{22-16}{9-3}=1$		
7	466		$\frac{228-118}{9-4}=22$			
		$\frac{922-466}{9-7}=228$				
9	922					

By Newton's divided difference formula

$$f(x) = f(x_0) + (x - x_0) \Delta f(x_0) + (x - x_0)(x - x_1) \Delta^2 f(x_0) + (x - x_0)(x - x_1)(x - x_2) \Delta^3 f(x_0) + \cdots$$

$$= 4 + (x - 0) 11 + (x - 0)(x - 2)(7) + (x - 0)(x - 2)(x - 3)(1)$$

$f(x) = x^3 + 2x^2 + 3x + 4$

$f'(x) = 3x^2 + 4x + 3$

$f''(x) = 6x + 4$

$f'(5) = 3(5)^2 + 4(5) + 3 = 98$

$f''(5) = 6(5) + 4 = 34$

$f(x)$ is maximum if $f'(x) = 0$

i.e., $3x^2 + 4x + 3 = 0$

But the roots of this eqn. are imaginary.

Hence there is no extremum value.

H.W. J. From the following table, find the value of x for which y is minimum and find the value of y.

x:	-2	-1	0	1	2	3	4
y:	2	-0.25	0	-0.25	2	15.75	56