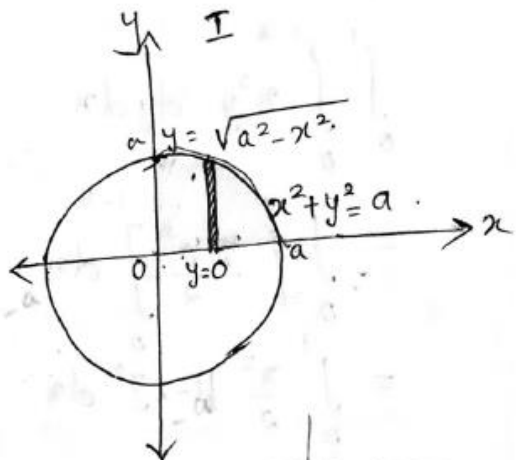
1) Evaluate $\iint_R xy\,dx\,dy$, R is the first quadrant of the circle $x^2+y^2=a^2$ $(x\geq 0,\ y\geq 0)$.

$$\int_0^a \int_0^{\sqrt{a^2-x^2}} xy\,dy\,dx$$



$$= \int_0^a \left. \frac{xy^2}{2} \right]_0^{\sqrt{a^2-x^2}} dx$$

$$= \int_0^a \frac{x}{2}\left[a^2-x^2\right] dx$$

$$= \frac{1}{2}\int_0^a (a^2x - x^3)\,dx$$

$$= \frac{1}{2}\left[\frac{a^2x^2}{2} - \frac{x^4}{4}\right]_0^a$$

$$= \frac{1}{2}\left[\frac{a^4}{2} - \frac{a^4}{4}\right] = \frac{a^4}{8}$$

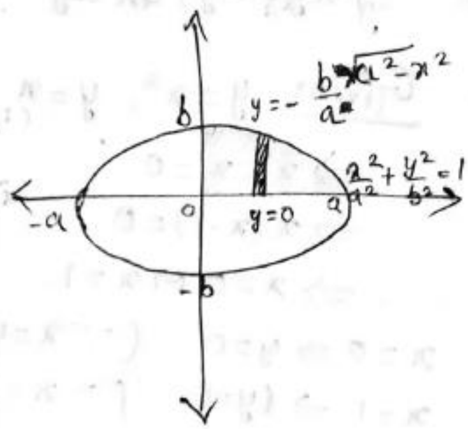
② $\iint_R x^2 y\, dy\, dx$ over the +ve quadrant of ellipse

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2} = \frac{a^2 - x^2}{a^2}$

$y^2 = \frac{b^2}{a^2}(a^2 - x^2)$

$y = \frac{b}{a}\sqrt{a^2 - x^2}$



$$= \int_0^a \int_0^{\frac{b}{a}\sqrt{a^2-x^2}} x^2 y\, dy\, dx$$

$$= \int_0^a \left[x^2 \frac{y^2}{2} \right]_0^{\frac{b}{a}\sqrt{a^2-x^2}} dx$$

$$= \int_0^a \frac{x^2}{2} \frac{b^2}{a^2} \left[a^2 - x^2\right] dx.$$

$$= \frac{b^2}{2a^2} \int_0^a (a^2x^2 - x^4)\, dx$$

$$= \frac{b^2}{2a^2} \left[a^2 \frac{x^3}{3} - \frac{x^5}{5} \right]_0^a = \frac{b^2}{2a^2} \left[\frac{a^5}{3} - \frac{a^5}{5} \right] = \frac{b^2}{\cancel{2}a^2} \times \frac{\cancel{2}a^{\cancel{5}\,3}}{15} = \frac{a^3 b^2}{15}$$

3) $\iint x^2y\,dx\,dy$ over the region in the +ve quadrant in which $x+y \le 1$.

$$\int_0^1 \int_0^{1-x} x^2y\,dy\,dx$$

$$= \int_0^1 \left[\frac{x^2y^2}{2}\right]_0^{1-x} dx$$

$$= \int_0^1 \frac{x^2}{2}[1-x]^2\,dx$$

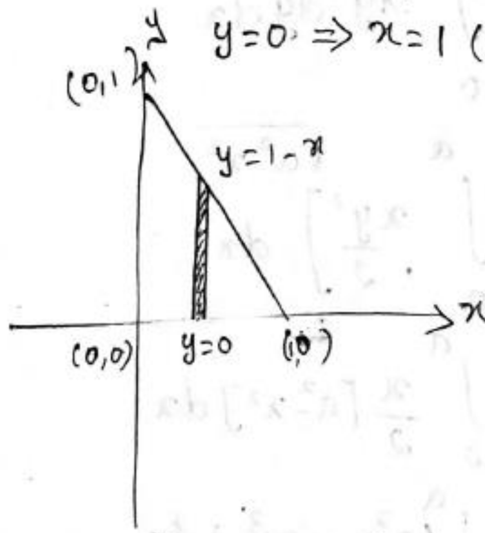
$$= \int_0^1 \frac{x^2}{2}[1+x^2-2x]\,dx$$

$$= \frac{1}{2}\int_0^1 (x^2+x^4-2x^3)\,dx$$

$$= \frac{1}{2}\left[\frac{x^3}{3}+\frac{x^5}{5}-\frac{\cancel{2}x^4}{\cancel{4}2}\right]_0^1$$

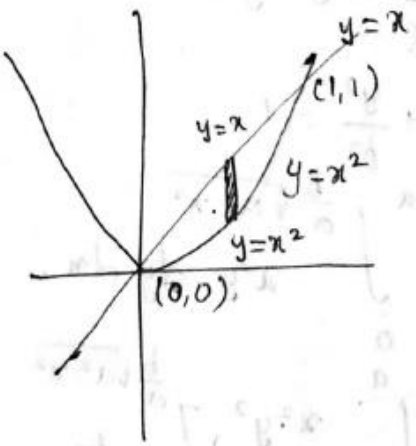
$$= \frac{1}{2}\left[\frac{1}{3}+\frac{1}{5}-\frac{1}{2}\right] = \frac{1}{60}$$

$x+y=1$
$x=0 \Rightarrow y=1$ $(0,1)$
$y=0 \Rightarrow x=1$ $(1,0)$



(4) Evaluate $\iint_R xy(x+y)\,dx\,dy$ over the region for which $y = x^2$, $y = x$.

$\int xy(x+y)\,dx\,dy$.



<u>Given:</u> $y = x^2$; $y = x$

$\Rightarrow x^2 - x = 0$

$\Rightarrow x(x-1) = 0$

$\Rightarrow x = 0$ or $x = 1$

$x = 0 \Rightarrow y = 0 \quad (\because x = y)$

$x = 1 \Rightarrow y = 1 \quad [\because x = y]$

$$\int_0^1 \int_{x^2}^{x} xy(x+y)\,dy\,dx$$

$$= \int_0^1 \int_{x^2}^{x} (x^2y + xy^2)\,dy\,dx$$

$$= \int_0^1 \left[\frac{x^2y^2}{2} + \frac{xy^3}{3} \right]^{x} dx$$

$$= \int_0^1 \left[\frac{x^4}{2} + \frac{x^4}{3} - \left[\frac{x^6}{2} + \frac{x^7}{3} \right] \right] dx$$

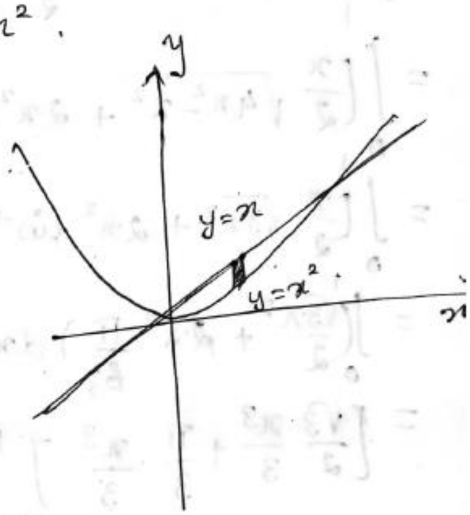
$$= \int_0^1 \frac{5x^4}{6} - \frac{x^6}{2} - \frac{x^7}{3} \, dx$$

$$= \left[\frac{5x^5}{30} - \frac{x^7}{14} - \frac{x^8}{24} \right]_0^1$$

$$= \frac{1}{6} - \frac{1}{14} - \frac{1}{24}$$

$$= \frac{9}{168}$$

(9) Evaluate $\iint_R (x-y)\,dx\,dy$ over the region between the line $x=y$ & parabola $y=x^2$.



$$\iint_R (x-y)\,dx\,dy$$

$\Rightarrow$ Given: $y=x^2$ & $y=x$

$\Rightarrow x^2 - x = 0$

$\Rightarrow x(x-1)=0$

$\Rightarrow x=0$ or $x=1$.

$x=0 \Rightarrow y=0 \quad [\because x=y]$

$x=1 \Rightarrow y=1 \quad [\because x=y]$

$$\int_0^1\int_{x^2}^{x} [x-y]\,dy\,dx.$$

$$= \int_0^1 \left[xy - \frac{y^2}{2}\right]_{x^2}^{x} dx$$

$$= \int_0^1 x^2 - \frac{x^2}{2} - \left[x^3 - \frac{x^4}{2}\right] dx$$

$$= \int_0^1 \left(\frac{x^2}{2} - x^3 + \frac{x^4}{2}\right) dx$$

$$= \left[\frac{x^3}{6} - \frac{x^4}{4} + \frac{x^5}{10}\right]_0^1 = \frac{1}{6} - \frac{1}{4} + \frac{1}{10} = \frac{40-60+24}{240}$$

$$= \frac{4}{240} = \frac{1}{60}$$