Eigen values of a matrix by Power method

Power method is used to determine the numerically largest eigen value and the corresponding eigen vector of a matrix A.

For every square matrix A, there is a scalar λ and a non-zero column vector X such that $AX = \lambda X$. Then the scalar λ is called as eigen value of A and X is the corresponding eigen vector.

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1. Find the dominant eigen value of $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ by Power method.

Soln.

Let $x_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$AX_0 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1+0 \\ 3+0 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} = 3\begin{pmatrix} 1/3 \\ 1 \end{pmatrix}$$

$$= 3\begin{pmatrix} 0.3333 \\ 1 \end{pmatrix} = 3X_1$$

$$AX_1 = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 0.3333 \\ 1 \end{pmatrix} = \begin{pmatrix} 0.3333+2 \\ 0.9999+4 \end{pmatrix} = \begin{pmatrix} 2.3333 \\ 4.9999 \end{pmatrix}$$

$$= 4.9999\begin{pmatrix} 0.4667 \\ 1 \end{pmatrix} = 4.9999\, x_2$$

$$AX_2 = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 0.4667 \\ 1 \end{pmatrix} = \begin{pmatrix} 0.4667+2 \\ 1.4001+4 \end{pmatrix} = \begin{pmatrix} 2.4667 \\ 5.4001 \end{pmatrix}$$

$$= 5.4001\begin{pmatrix} 0.4568 \\ 1 \end{pmatrix} = 5.4001\, X_3$$

$$AX_3 = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 0.4568 \\ 1 \end{pmatrix} = \begin{pmatrix} 0.4568+2 \\ 1.3704+4 \end{pmatrix} = \begin{pmatrix} 2.4568 \\ 5.3704 \end{pmatrix}$$

$$= 5.3704\begin{pmatrix} 0.4575 \\ 1 \end{pmatrix} = 5.3704\, X_4$$

$$\therefore \; AX_4 = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 0.4575 \\ 1 \end{pmatrix} = \begin{pmatrix} 0.4575+2 \\ 1.3725+4 \end{pmatrix} = \begin{pmatrix} 2.4575 \\ 5.3725 \end{pmatrix}$$

$$= 5.3725 \begin{pmatrix} 0.4574 \\ 1 \end{pmatrix} = 5.3725 \, X_5$$

$$AX_5 = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 0.4574 \\ 1 \end{pmatrix} = \begin{pmatrix} 0.4574+2 \\ 1.3722+4 \end{pmatrix} = \begin{pmatrix} 2.4574 \\ 5.3722 \end{pmatrix}$$

$$= 5.3722 \begin{pmatrix} 0.4574 \\ 1 \end{pmatrix} = 5.3722 \, X_6$$

$$AX_6 = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 0.4574 \\ 1 \end{pmatrix} = \begin{pmatrix} 0.4574+2 \\ 1.3722+4 \end{pmatrix} = \begin{pmatrix} 2.4574 \\ 5.3722 \end{pmatrix}$$

$$= 5.3722 \begin{pmatrix} 0.4574 \\ 1 \end{pmatrix} = 5.3722 \, X_7$$

$\therefore$ Eigen value $\lambda_1 = 5.3722$

Corresponding Eigen vector $X_1 = \begin{pmatrix} 0.4574 \\ 1 \end{pmatrix}$

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(2). Find the numerically largest eigen value of the matrix $A = \begin{pmatrix} 3 & -5 \\ -2 & 4 \end{pmatrix}$ by power method. find the other eigen value also.

Soln.

Let $X_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Soln.

$$A X_0 = \begin{pmatrix} 3 & -5 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3+0 \\ -2+0 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ -0.6667 \end{pmatrix}$$

$$= 3X_1$$

$$AX_1 = \begin{pmatrix} 3 & -5 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ -0.6667 \end{pmatrix} = \begin{pmatrix} 3+3.3335 \\ -2-2.6668 \end{pmatrix} = \begin{pmatrix} 6.3335 \\ -4.6668 \end{pmatrix}$$

$$= 6.3335 \begin{pmatrix} 1 \\ -0.7368 \end{pmatrix} = 6.3335 \, X_2$$

$$A x_2 = \begin{pmatrix} 3 & -5 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ -0.7368 \end{pmatrix} = \begin{pmatrix} 3+3.6840 \\ -2-2.9472 \end{pmatrix} = \begin{pmatrix} 6.6840 \\ -4.9472 \end{pmatrix}$$

$$= 6.6840 \begin{pmatrix} 1 \\ -0.7402 \end{pmatrix} = 6.6840\, x_3$$

$$A x_3 = \begin{pmatrix} 3 & -5 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ -0.7402 \end{pmatrix} = \begin{pmatrix} 3+3.7010 \\ -2-2.9608 \end{pmatrix} = \begin{pmatrix} 6.7010 \\ -4.9608 \end{pmatrix}$$

$$= 6.7010 \begin{pmatrix} 1 \\ -0.7403 \end{pmatrix} = 6.7010\, x_4$$

$$A x_4 = \begin{pmatrix} 3 & -5 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ -0.7403 \end{pmatrix} = \begin{bmatrix} 3+3.7015 \\ -2-2.9612 \end{bmatrix} = \begin{pmatrix} 6.7015 \\ -4.9612 \end{pmatrix}$$

$$= 6.7015 \begin{pmatrix} 1 \\ -0.7403 \end{pmatrix} = 6.7015\, x_5$$

$$A x_5 = \begin{pmatrix} 3 & -5 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ -0.7403 \end{pmatrix} = \begin{pmatrix} 3+3.7015 \\ -2-2.9612 \end{pmatrix} = \begin{pmatrix} 6.7015 \\ -4.9612 \end{pmatrix}$$

$$= 6.7015 \begin{pmatrix} 1 \\ -0.7403 \end{pmatrix} = 6.7015\, x_6$$

$\therefore$ Eigen value $\lambda_1 = 6.7015$

corresponding Eigen vector $X_1 = \begin{pmatrix} 1 \\ -0.7403 \end{pmatrix}$

ii). To find the other Eigen value:

$\because$ Sum of the eigen values = Sum of the diagonal elemen

$$\lambda_1 + \lambda_2 = 3 + 4$$

$$6.7015 + \lambda_2 = 7$$

$$\lambda_2 = 7 - 6.7015$$

$$\lambda_2 = 0.2984$$

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③ Find the dominant eigen value and the corresponding eigen vector of $A = \begin{pmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

3 −5
−2 4
6.7015 $\begin{pmatrix} 1 \\ -0.7403 \end{pmatrix}$

1 3 −1
3 2 4
−1 4 10

11.663
$\begin{pmatrix} 0.025 \\ 0.422 \\ 1 \end{pmatrix}$

Soln. :

Let $x_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$$Ax_0 = \begin{pmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1+0+0 \\ 1+0+0 \\ 0+0+0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = 1\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = 1\,x_1$$

$$Ax_1 = \begin{pmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1+6+0 \\ 1+2+0 \\ 0+0+0 \end{pmatrix} = \begin{pmatrix} 7 \\ 3 \\ 0 \end{pmatrix} = 7\begin{pmatrix} 1 \\ 0.4286 \\ 0 \end{pmatrix} = 7\,x_2$$

$$Ax_2 = \begin{pmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 0.4286 \\ 0 \end{pmatrix} = \begin{pmatrix} 1+2.5716+0 \\ 1+0.8572+0 \\ 0+0+0 \end{pmatrix} = \begin{pmatrix} 3.5716 \\ 1.8572 \\ 0 \end{pmatrix}$$

$$= 3.5716\begin{pmatrix} 1 \\ 0.52 \\ 0 \end{pmatrix} = 3.5716\,x_3$$

$$Ax_3 = \begin{pmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 0.52 \\ 0 \end{pmatrix} = \begin{pmatrix} 1+3.1199+0 \\ 1+1.04+0 \\ 0+0+0 \end{pmatrix} = \begin{pmatrix} 4.1199 \\ 2.04 \\ 0 \end{pmatrix}$$

$$= 4.1199\begin{pmatrix} 1 \\ 0.4952 \\ 0 \end{pmatrix} = 4.1199\,x_4$$

$$Ax_4 = \begin{pmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 0.4952 \\ 0 \end{pmatrix} = \begin{pmatrix} 1+2.9709+0 \\ 1+0.9904+0 \\ 0+0+0 \end{pmatrix} = \begin{pmatrix} 3.9709 \\ 1.9904 \\ 0 \end{pmatrix}$$

$$= 3.9709\begin{pmatrix} 1 \\ 0.5012 \\ 0 \end{pmatrix} = 3.9709\,x_5$$

$$Ax_5 = \begin{pmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 0.5012 \\ 0 \end{pmatrix} = \begin{pmatrix} 1+3.0072+0 \\ 1+1.0024+0 \\ 0+0+0 \end{pmatrix} = \begin{pmatrix} 4.0072 \\ 2.0024 \\ 0 \end{pmatrix}$$

$$= 4.0072\begin{pmatrix} 1 \\ 0.4997 \\ 0 \end{pmatrix} = 4.0072\,x_6$$

$$Ax_6 = \begin{pmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 0.4997 \\ 0 \end{pmatrix} = \begin{pmatrix} 1+2.9982+0 \\ 1+0.9994+0 \\ 0+0+0 \end{pmatrix} = \begin{pmatrix} 3.9982 \\ 1.9994 \\ 0 \end{pmatrix}$$

$$= 3.9982\begin{pmatrix} 1 \\ 0.5001 \\ 0 \end{pmatrix} = 3.9982\,x_7$$

$$Ax_7 = \begin{pmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0.5001 \\ 0 \end{pmatrix} = \begin{pmatrix} 1+3.005+0 \\ 1+1.0002+0 \\ 0+0+0 \end{pmatrix} = \begin{pmatrix} 4.005 \\ 2.0002 \\ 0 \end{pmatrix}$$

$$= 4.005 \begin{pmatrix} 1 \\ 0.4994 \\ 0 \end{pmatrix} = 4.005\, x_8$$

$$Ax_8 = \begin{pmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0.4994 \\ 0 \end{pmatrix} = \begin{pmatrix} 1+2.9966+0 \\ 1+0.9988+0 \\ 0+0+0 \end{pmatrix} = \begin{pmatrix} 3.9966 \\ 1.9988 \\ 0 \end{pmatrix}$$

$$= 3.9966 \begin{pmatrix} 1 \\ 0.5001 \\ 0 \end{pmatrix} = 3.9966\, x_9$$

$$Ax_9 = \begin{pmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0.5001 \\ 0 \end{pmatrix} = \begin{pmatrix} 1+3.0006+0 \\ 1+1.0002+0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 4.0006 \\ 2.0002 \\ 0 \end{pmatrix} = 4.0006 \begin{pmatrix} 1 \\ 0.5 \\ 0 \end{pmatrix} = 4.0006\, x_{10}$$

$$Ax_{10} = \begin{pmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0.5 \\ 0 \end{pmatrix} = \begin{pmatrix} 1+3+0 \\ 1+1+0 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix} = 4 \begin{pmatrix} 1 \\ 0.5 \\ 0 \end{pmatrix}$$

$$= 4\, x_{11}$$

$$Ax_{11} = \begin{pmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0.5 \\ 0 \end{pmatrix} = \begin{pmatrix} 1+3+0 \\ 1+1+0 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix} = 4 \begin{pmatrix} 1 \\ 0.5 \\ 0 \end{pmatrix}$$

$\therefore$ Dominant eigen value $\lambda = 4$

$$x = \begin{pmatrix} 1 \\ 0.5 \\ 0 \end{pmatrix}$$

HW Find the dominant eigen value & the corresponding eigen vector of the matrix

$$A = \begin{bmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{bmatrix}$$

Ans. $\lambda = 11.66$

$$x = \begin{pmatrix} 0.025 \\ 0.421 \\ 1 \end{pmatrix}$$