Wiener - Khintchine Theorem :

If $X_T(\omega)$ is the FT of the truncated random process is defined as,

$$X_T(t) = \begin{cases} X(t) & \text{for } |t| \leq T \\ 0 & \text{for } |t| > T \end{cases}$$

where $[X(t)]$ is a real WSS process with power spectral density function $S_{XX}(\omega)$, then

$$S_{XX}(\omega) = \lim_{T \to \infty} \left[\frac{1}{2T} E\left\{ |X_T(\omega)|^2 \right\} \right]$$

Proof :

Given $X_T(\omega)$ is the Fourier transform of X_T

$$\Rightarrow \quad X_T(\omega) = \int_{-\infty}^{\infty} X_T(t)\, e^{-i\omega t}\, dt$$

$$= \int_{-T}^{T} x(t)\, e^{-i\omega t}\, dt$$

Now, $$|X_T(\omega)|^2 = [X_T(\omega)]\,[\overline{X_T(\omega)}] \qquad \because |z|^2 = z\bar{z}$$

$$= X_T(\omega)\, X_T(-\omega)$$

$$= \int_{-T}^{T} x(t_1)\, e^{-i\omega t_1}\, dt_1 \cdot \int_{-T}^{T} x(t_2)\, e^{i\omega t_2}\, dt_2$$

$$= \int_{-T}^{T}\int_{-T}^{T} x(t_1)\, x(t_2)\, e^{-i\omega(t_1 - t_2)}\, dt_1\, dt_2$$

$$E[|X_T(\omega)|^2] = \int_{-T}^{T}\int_{-T}^{T} E[x(t_1)\, x(t_2)]\, e^{-i\omega(t_1 - t_2)}\, dt$$

= a function of $(t_1 - t_2)$ $\{x(t)$ is a

$$= \int_{-T}^{T}\int_{-T}^{T} R_{XX}(t_1-t_2)\, e^{-i\omega(t_1-t_2)}\, dt_1\, dt_2$$

$$= \int_{-T}^{T}\int_{-T}^{T} R_{XX}(t_1-t_2)\, e^{-i\omega(t_1-t_2)}\, dt_1\, dt_2$$

Let $t_2 = t \Rightarrow \tau = t_1 - t \Rightarrow t_1 = \tau + t$

$dt_2 = dt$ $\qquad dt_1 = d\tau$

when $t_1 = -T \Rightarrow \tau = -T - t$

$t_1 = T \Rightarrow \tau = T - t$

when $t_2 = -T \Rightarrow t = -T$

$t_2 = T \Rightarrow t = T$

$$E[|X_T(\omega)|^2] = \int_{-T}^{T}\int_{-T-t}^{T-t} R_{XX}(\tau)\, e^{-i\omega(\tau)}\, d\tau\, dt$$

$$= \int_{-T}^{T} dt \int_{-T-t}^{T-t} R_{XX}(\tau)\, e^{-i\omega\tau}\, d\tau$$

$$= 2T \int_{-T-t}^{T-t} R_{XX}(\tau)\, e^{-i\omega\tau}\, d\tau$$

$$\lim_{T\to\infty} \frac{1}{2T} E[|X_T(\omega)|^2] = \lim_{T\to\infty} \frac{1}{2T}\, 2T \int_{-T-t}^{T-t} R_{XX}(\tau)\, e^{-i\omega\tau}\, d\tau$$

$$= \lim_{T\to\infty} \int_{-T-t}^{T-t} R_{XX}(\tau)\, e^{-i\omega\tau}\, d\tau$$

$$= \int_{-\infty}^{\infty} R_{XX}(\tau)\, e^{-i\omega\tau}\, d\tau = S_{XX}(\omega)$$

Hence proved.