Cross Spectral Density (or) Cross Power Spectral density

Consider two jointly WSS random processes $X(t)$ and $Y(t)$.

Let $R_{XY}(\tau)$ and $R_{YX}(\tau)$ be their cross correlation functions.

Then the cross spectral densities are,

$$S_{XY}(\omega) = \int_{-\infty}^{\infty} R_{XY}(\tau)\, e^{-i\omega\tau}\, d\tau$$

$$S_{YX}(\omega) = \int_{-\infty}^{\infty} R_{YX}(\tau)\, e^{-i\omega\tau}\, d\tau$$

Properties :

I. $S_{xy}(\omega) = S_{yx}(-\omega)$

Proof :

Now, $S_{xy}(\omega) = \int_{-\infty}^{\infty} R_{xy}(\tau)\, e^{-i\omega\tau}\, d\tau$

$= \int_{-\infty}^{\infty} R_{yx}(-\tau)\, e^{-i\omega\tau}\, d\tau$

$\because R_{xy}(\tau) = R_{yx}(-\tau)$

$= \int_{\infty}^{-\infty} R_{yx}(\tau_1)\, e^{i\omega\tau_1}\, (-d\tau_1)$ | put $\tau_1 = -\tau$, when $\tau = \infty \Rightarrow \tau_1 = -\infty$, $\tau = -\infty \Rightarrow \tau_1 = \infty$

$= \int_{-\infty}^{\infty} R_{yx}(\tau_1)\, e^{i\omega\tau_1}\, d\tau_1$

$= S_{yx}(-\omega)$

$\therefore S_{xy}(\omega) = S_{yx}(-\omega)$

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2. Real part of $S_{xy}(\omega)$ is an even function of ω. ie, $S_{xy}(\omega) = S_{xy}(-\omega)$

Proof :

$S_{xy}(\omega) = \int_{-\infty}^{\infty} R_{xy}(\tau)\, e^{-i\omega\tau}\, d\tau$

$= \int_{-\infty}^{\infty} R_{xy}(\tau)\,(\cos\omega\tau - i\sin\omega\tau)\, d\tau$

Real part of $S_{xy}(\omega) = \int_{-\infty}^{\infty} R_{xy}(\tau)\cos\omega\tau\, d\tau$

$$Re\,[S_{xy}(-\omega)] = \int_{-\infty}^{\infty} R_{xy}(\tau) \cos(-\omega)\tau \, d\tau$$

$$= \int_{-\infty}^{\infty} R_{xy}(\tau) \cos \omega\tau \, d\tau$$

$$= Re\,[S_{xy}(\omega)] \quad \because \cos(-\theta) = \cos\theta$$

$\therefore Re\,[S_{xy}(\omega)]$ is an even function of ω.

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3. Imaginary part of $S_{xy}(\omega)$ is an odd function of ω.

proof :

$$S_{xy}(\omega) = \int_{-\infty}^{\infty} R_{xy}(\tau)\, e^{-i\omega\tau} \, d\tau$$

$$= \int_{-\infty}^{\infty} R_{xy}(\tau) (\cos \omega\tau - i \sin \omega\tau) \, d\tau$$

$$Im\,[S_{xy}(\omega)] = -\int_{-\infty}^{\infty} R_{xy}(\tau) \sin \omega\tau \, d\tau \longrightarrow (1)$$

$$Im\,[S_{xy}(-\omega)] = -\int_{-\infty}^{\infty} R_{xy}(\tau) \sin(-\omega\tau) \, d\tau$$

$$= \int_{-\infty}^{\infty} R_{xy}(\tau) \sin(\omega\tau) \, d\tau$$

$$[\because \sin(-\theta) = -\sin\theta]$$

$$= -Im\,[S_{xy}(\omega)] \quad [\text{By } (1)]$$

$\therefore Im[S_{xy}(\omega)]$ is an odd function of ω.

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J. The cross power spectrum of a real random process $X(t)$ and $Y(t)$ is given by,

$$S_{XY}(\omega) = \begin{cases} a + \frac{ib\omega}{\alpha}, & -\alpha < \omega < \alpha, \alpha > 0 \\ 0, & \text{otherwise.} \end{cases}$$

Find the cross correlation function.

Soln.

$$R_{XY}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XY}(\omega) e^{i\omega\tau} d\omega$$

$$= \frac{1}{2\pi} \int_{-\alpha}^{\alpha} \left(a + \frac{ib\omega}{\alpha}\right) e^{i\omega\tau} d\omega$$

$$= \frac{1}{2\pi} \int_{-\alpha}^{\alpha} a e^{i\omega\tau} d\omega + \frac{1}{2\pi} \int_{-\alpha}^{\alpha} \frac{ib\omega}{\alpha} e^{i\omega\tau} d\omega$$

$$= \frac{a}{2\pi} \left[\frac{e^{i\omega\tau}}{i\tau}\right]_{-\alpha}^{\alpha} + \frac{ib}{2\pi\alpha} \left[\omega\left(\frac{e^{i\omega\tau}}{i\tau}\right) - \left(\frac{e^{i\omega\tau}}{(i\tau)^2}\right)\right]_{-\alpha}^{\alpha}$$

$$= \frac{a}{2\pi i\tau} \left[e^{i\alpha\tau} - e^{-i\alpha\tau}\right] + \frac{ib}{2\pi\alpha} \left[\left(\frac{\alpha}{i\tau} e^{i\alpha\tau} + \frac{e^{i\alpha\tau}}{\tau^2}\right) - \left(-\frac{\alpha e^{-i\alpha\tau}}{i\tau} + \frac{e^{-i\alpha\tau}}{\tau^2}\right)\right]$$

$$= \frac{a}{2\pi i\tau} [2i \sin \alpha\tau] + \frac{ib}{2\pi\alpha} \left[\frac{\alpha}{i\tau} (e^{i\alpha\tau} - e^{-i\alpha\tau}) + \frac{1}{\tau^2} (e^{i\alpha\tau} - e^{-i\alpha\tau})\right]$$

$$= \frac{a}{\pi\tau} \sin \alpha\tau + \frac{ib}{2\pi\alpha} \left[\frac{\alpha}{i\tau} 2 \cos \alpha\tau + \frac{1}{\tau^2} 2i \sin \alpha\tau\right]$$

$$= \frac{a}{\pi\tau} \sin \alpha\tau + \frac{b}{\pi\tau} \cos \alpha\tau - \frac{b}{\pi\alpha\tau^2} \sin \alpha\tau$$

2]. If the cross correlation of two processes $\{X(t)\}$ and $\{Y(t)\}$ is $R_{xy}(t, t+\tau) = \frac{AB}{2}[\sin(\omega_0\tau) + \cos(\omega_0(2t+\tau))]$

where A, B are ω_0 and constants. Find the cross power spectrum.

Soln.

The time average is given by,

$$R_{xy}(\tau) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} R_{xy}(t, t+\tau)\, d\tau$$

$$= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \frac{AB}{2}[\sin\omega_0\tau + \cos\omega_0(2t+\tau)]\, d\tau$$

$$= \frac{AB}{2}\left[\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \sin(\omega_0\tau)\, d\tau + \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} \cos\omega_0(2t+\tau)\, d\tau\right]$$

$$= \frac{AB}{2}\left[\sin\omega_0\tau + \lim_{T\to\infty}\frac{1}{4T}\sin(\omega_0(2t+\tau))\right]_{-T}^{T}$$

$$= \frac{AB}{2}\sin\omega_0 t + 0$$

The cross spectrum is

$$S_{xy}(\omega) = \int_{-\infty}^{\infty} R_{xy}(\tau)\, e^{-i\omega\tau}\, d\tau$$

$$= \int_{-\infty}^{\infty} \frac{AB}{2}\sin\omega_0\tau\, e^{-i\omega\tau}\, d\tau$$

$$= \frac{AB}{2}\int_{-\infty}^{\infty} \sin\omega_0\tau(\cos\omega\tau - i\sin\omega\tau)\, d\tau$$

$$= \frac{AB}{2}\left[\int_{-\infty}^{\infty} \sin \omega_0 \tau \cos \omega \tau \, d\tau - i \int_{-\infty}^{\infty} \sin \omega_0 \tau \sin \omega \tau \, d\tau\right]$$

$$= \frac{AB}{2}\left[\frac{1}{2}\int_{-\infty}^{\infty}\left[\sin(\omega_0+\omega)\tau + \sin(\omega_0-\omega)\tau\right] d\tau - \frac{i}{2}\int_{-\infty}^{\infty}\left[\cos(\omega-\omega_0)\tau - \cos(\omega+\omega_0)\tau\right] d\tau\right]$$

$$= \frac{AB}{4}\left[\int_{-\infty}^{\infty} \sin(\omega_0+\omega)\tau + \sin(\omega_0-\omega)\tau - i\cos(\omega-\omega_0)\tau + i\cos(\omega+\omega_0)\tau\right] d\tau\Big]$$

$$= \frac{AB}{4}\int_{-\infty}^{\infty}\left\{i\left[\cos(\omega-\omega_0)\tau - i\sin(\omega-\omega_0)\tau\right] - i\left[\cos(\omega-\omega_0)\tau - i\sin(\omega-\omega_0)\tau\right]\right\} d\tau$$

$$= -\frac{iAB}{4}\int_{-\infty}^{\infty}\left[e^{-i(\omega-\omega_0)\tau} - e^{-i(\omega+\omega_0)\tau}\right] d\tau$$

$$= -\frac{iAB}{4}\left[\int_{-\infty}^{\infty} e^{-i(\omega-\omega_0)\tau} d\tau - \int_{-\infty}^{\infty} e^{-i(\omega+\omega_0)\tau} d\tau\right]$$

$$= -\frac{iAB}{4}\left[2\pi\delta(\omega-\omega_0) - 2\pi\delta(\omega+\omega_0)\right]$$

$$= -\frac{i\pi AB}{2}\left[\delta(\omega-\omega_0) - \delta(\omega+\omega_0)\right]$$

3. Given the Power Spectral density $S_{XX}(\omega) = \frac{\omega^2+9}{\omega^4+5\omega^2+4}$. find the mean square value of the process.

Soln.

Given $S_{XX}(\omega) = \frac{\omega^2+9}{\omega^4+5\omega^2+4}$

put $u = \omega^2$

$$S_{XX}(\omega) = \frac{u+9}{u^2+5u+4} = \frac{u+9}{(u+1)(u+4)} = \frac{A}{u+1} + \frac{B}{u+4}$$

$$u+9 = A(u+4) + B(u+1)$$

put $u=-1$, $-1+9 = A(-1+4)+0 \Rightarrow A = 8/3$

$u=-4$, $-4+9 = 0+B(-4+1) \Rightarrow B = -5/3$

$$\therefore S_{XX}(\omega) = \frac{8}{3}\frac{1}{u+1} - \frac{5}{3}\frac{1}{u+4} = \frac{8}{3}\frac{1}{\omega^2+1} - \frac{5}{3}\frac{1}{\omega^2+4}$$

To find :

mean square value of the process i.e., $R_{XX}(0)$

WKT

$$R_{XX}(\tau) = F^{-1}(S_{XX}(\omega))$$

$$= F^{-1}\left[\frac{8}{3}\frac{1}{\omega^2+1} - \frac{5}{3}\frac{1}{\omega^2+4}\right]$$

$$= \frac{8}{3}\cdot\frac{1}{2}F^{-1}\left(\frac{2}{\omega^2+1^2}\right) - \frac{5}{3}\cdot\frac{1}{4}F^{-1}\left(\frac{2\cdot 2}{\omega^2+2^2}\right)$$

$$= \frac{8}{6}e^{-|\tau|} - \frac{5}{12}e^{-2|\tau|}$$

Now, $R_{XX}(0) = \frac{8}{6} - \frac{5}{12}$

$$= \frac{11}{12}$$