

UNIT – 1

Two marks

1. Define i) Discrete random variable

ii) Continuous random variable

i) Let  $X$  be a random variable, if the number of possible values of  $X$  is finite or count ably finite, then  $X$  is called a discrete random variable.

ii) A random variable  $X$  is called the continuous random variable, if  $x$  takes all its possible values in an interval.

2. Define probability mass function (PMF):

Let  $X$  be the discrete random variable taking the values  $,X_1 , X_2 \dots\dots\dots$

Then the number  $P (X_i) = P(X =X_i )$  is called the probability mass function of  $X$  and it satisfies the following conditions.

i)  $P (X_i) \geq 0$  for all;

ii)  $\sum_{i=1}^{\infty} P (X_i) = 1$

3. Define probability Density function (PDF):

Let  $x$  be a continuous random variable. The Function  $f(x)$  is called the probability density function (PDF) of the random variable  $x$  if it satisfies.

i)  $f(x) \geq 0$

ii)  $\int_{-\infty}^{\infty} f(x)dx = 1$

4. Define cumulative distribution function (CDF):

Let  $x$  be a random variable. The cumulative distribution function, denoted by  $F(X)$ and is given by  $F(X)=P(X\leq x)$

5. If  $x$  is a discrete R.V having the p.m.f

X:	-1	0	1
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## PROBABILITY AND RANDOM PROCESSES

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P(X):	k	2k	3k
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Find  $P(x \geq 0)$

**Answer:**  $6k = 1 \Rightarrow k = \frac{1}{6}$

$$P[x \geq 0] = 2k + 3k \Rightarrow P[x \geq 0] = \frac{1}{6}$$

6. The random variable  $x$  has the p.m.f.  $P(x) = \frac{x}{15}$ ,  $x=1,2,3,4,5$  and  $= 0$  else where.

Find  $P\left[\frac{1}{2} < x < \frac{5}{2} / x > 1\right]$ . **Answer:**

$$P\left[\frac{1}{2} < x < \frac{5}{2} / x > 1\right] = \frac{P[x=2]}{P(x>1)} = \frac{P[x=2]}{1-P(x \leq 1)} = \frac{2/15}{1-1/15} = \frac{1}{7}$$

7. If the probability distribution of  $X$  is given as :

X	1	2	3	4
P(X)	0.4	0.3	0.2	0.1

Find  $P\left[\frac{1}{2} < x < \frac{7}{2} / x > 1\right]$ .

**Answer :**

$$P\left[\frac{1}{2} < x < \frac{7}{2} / x > 1\right] = \frac{P[1 < x < 7/2]}{P(x > 1)} = \frac{P(x=2)+P(x=3)}{1-P(x=1)} = \frac{0.5}{0.6} = \frac{5}{6}$$

8. A.R.V.  $X$  has the probability function

X	-2	-1	0	1
P(X)	0.4	k	0.2	0.3

Find  $k$  and the mean value of  $X$

**Answer:**

$$k=0.1 \quad \text{Mean} = \sum xP(x) = \frac{1}{10} [-8-1+0+3] = -0.6$$

9. If the p.d.f of a R.V.  $X$  is  $f(x) = \frac{x}{2}$  in  $0 \leq x \leq 2$ , find

$$P[x > 1.5/x > 1].$$

**Answer :**

$$P[x > 1.5/x > 1] = \frac{P[x > 1.5]}{P(x > 1)} = \frac{\int_{1.5}^2 \frac{x}{2} dx}{\int_1^2 \frac{x}{2} dx} = \frac{4 - 2.25}{4 - 1} = 0.5833$$

10. If the p.d.f of a R.V. X is given by  $f(x) = \{1/4, -2 < x < 2.0, \text{ else where. Find } P[|X| > 1]$ .

**Answer:**

$$P[|X| > 1] = 1 - P[|X| < 1] = 1 - \int_{-1}^1 \frac{1}{4} dx = \frac{1}{2}$$

11. If  $f(x) = kx^2, 0 < x < 3$  is to be density function, Find the value of k.

**Answer:**

$$\int_0^3 kx^2 dx = 1 \Rightarrow 9k = 1 \therefore k = \frac{1}{9}$$

12. If the c.d.f. of a R.V X is given by  $F(x) = 0$  for  $x < 0$ ;  $= \frac{x^2}{16}$  for  $0 \leq x < 4$  and  $= 1$  for  $x \geq 4$ , find  $P(X > 1/X < 3)$ .

**Answer:**

$$P(X > 1/X < 3) = \frac{P[1 < X < 3]}{P[0 < X < 3]} = \frac{F(3) - F(1)}{F(3) - F(0)} = \frac{8/16}{9/16} = \frac{8}{9}$$

13. The cumulative distribution of X is  $F(x) = \frac{x^3 + 1}{9}, -1 < X < 2$  and  $= 0$ , otherwise. Find  $P[0 < X < 1]$ .

**Answer:**

$$P[0 < X < 1] = F(1) - F(0) = \frac{2}{9} - \frac{1}{9} = \frac{1}{9}$$

14. A Continuous R.V X that can assume any value between  $x=2$  and  $x=5$  had the p.d.f  $f(x) = k(1+x)$ . Find  $P(x < 4)$ .

**Answer:**

$$\int_2^3 k(1+x)dx = 1 \Rightarrow \frac{27k}{2} = 1 \therefore k = \frac{2}{27}$$

$$P[X < 4] = \int_2^4 \frac{2}{27}(1+x)dx = \frac{16}{27}$$

15. The c.d.f of X is given by  $F(x) = \begin{cases} 0, & x > 1 \\ x^2, & 0 \leq x \leq 1 \\ 1, & x < 0 \end{cases}$  Find the p.d.f of x, and

obtain  $P(X > 0.75)$ .

**Answer:**

$$f(x) = \frac{d}{dx}F(x) = \begin{cases} 2x, & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

$$P[x < 0.75] = 1 - P[X \leq 0.75] = 1 - F(0.75) = 1 - (0.75)^2 = 0.4375$$

16. Check whether  $f(x) = \frac{1}{4}xe^{-x/2}$  for  $0 < x < \infty$  can be the p.d.f of X.

**Answer:**

$$= \int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} \frac{x}{4}e^{-x/2} dx = \int_0^{\infty} te^{-t} dt \text{ where } t = \frac{x}{2}$$

$$= (-te^{-t} - e^{-t})_0^{\infty} = -[0-1] = 1$$

$\therefore f(x)$  is the p.d.f of X.

17. A continuous R.V X has a p.d.f  $f(x) = 3x^2, 0 \leq x \leq 1$ . Find b such that

$P(X > b) = 0.05$ .

**Answer:**

$$3 \int_b^1 x^2 dx = 0.05 \Rightarrow 1 - b^3 = 0.05 \Rightarrow b^3 = 0.95 \therefore b = (0.95)^{\frac{1}{3}}$$

## PROBABILITY AND RANDOM PROCESSES

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18. Let  $X$  be a random variable taking values  $-1, 0$  and  $1$  such that  $P(X=-1) = 2P(X=0) = P(X=1)$ . Find the mean of  $2X-5$ .

**Answer:**

$$\sum P(X = x) = 1 \Rightarrow 5P(X = 0) = 1 \therefore P(X = 0) = \frac{1}{5}$$

Probability distribution of  $X$ :

$X$	$-1$	$0$	$1$
$P(X)$	$\frac{2}{5}$	$\frac{1}{5}$	$\frac{2}{5}$

$$\text{Mean} = E(x) = \sum xp(x) = -1\left(\frac{2}{5}\right) + 0\left(\frac{1}{5}\right) + 1\left(\frac{2}{5}\right) = 0$$

$$E[2X-5] = 2E(X) - 5 = 2[0] - 5 = -5.$$

19. Find the cumulative distribution function  $F(x)$  corresponding to the p.d.f.

$$f(x) = \frac{1}{\pi(1+x^2)}, \quad -\infty < x < \infty.$$

**Answer**

$$\begin{aligned} F(x) &= \int_{-\infty}^x f(x) dx = \frac{1}{\pi} \int_{-\infty}^x \frac{dx}{1+x^2} = \frac{1}{\pi} [\tan^{-1} x] \\ &= \frac{1}{\pi} \left[ \frac{\pi}{2} + \tan^{-1} x \right] \end{aligned}$$

20. The diameter of an electric cable, say  $X$  is assumed to a continuous R.V with

p.d.f of given by  $f(x) = kx(1-x)$ ,  $0 \leq x \leq 1$ . Determine  $k$  and  $P\left(x \leq \frac{1}{3}\right)$

**Answer:**

$$\int_0^1 kx(1-x) dx = 1 \Rightarrow k \left[ \frac{1}{2} - \frac{1}{3} \right] = 1 \therefore k = 6$$

$$P\left[X \leq \frac{1}{3}\right] = 6 \int_0^{1/3} (x - x^2) dx = 6 \left[ \frac{x^2}{2} - \frac{x^3}{3} \right]_0^{1/3} = [(3x^2 - 2x^3)]_0^{1/3} = \frac{1}{3} - \frac{2}{27} = \frac{7}{27}$$

21. A random variable X has the p.d.f  $f(x)$  given by  $f(x) = \begin{cases} Cxe^{-x}, & \text{if } x > 0 \\ 0, & \text{if } x \leq 0 \end{cases}$ . Find the value of C and C.D.F of X.

**Answer:**

$$C \int_0^{\infty} xe^{-x} dx = 1 \Rightarrow C[x(-e^{-x})]_0^{\infty} = 1$$

$$\therefore C[-0 + 1] = 1 \Rightarrow C = 1$$

C.D.F :  $F(x) = \int_0^x f(x) dx = 1 - (1+x)e^{-x}$  for  $x \geq 0$ .

22. State the properties of cumulative distribution function.

**Answer:**

- i)  $F(-\infty)=0$  and  $F(\infty) = 1$ .
- ii)  $F(\infty)$  is non – decreasing function of X.
- iii) If  $F(\infty)$  is the p. d. f of X, then  $f(x)=F'(x)$
- iv)  $P[a \leq X \leq b] = F(b) - F(a)$

23. Define the raw and central moments of R.V and state the relation between them.

**Answer:**

Raw moment  $\mu'_r = E[X^r]$

Central moment  $\mu_r = E[\{X - E(X)\}^r]$ .

$$\mu_r = \mu'_r - rC_1\mu'_{r-1} \mu'_r + rC_2\mu'_{r-2}(\mu'_r)^2 - \dots + (-1)^r(\mu'_1)^r$$

24. The first three moments of a R.V.X about 2 are 1, 16, -40. Find the mean, variance of X. Hence find  $\mu_3$ .

**Answer:**

$E(X) = \mu'_1 + A \Rightarrow \text{Mean} = 1 + 2 = 3$

Variance =  $E(X^2) - [E(X)]^2 = 16 - 1 = 15$

$$\mu_3 = \mu'_3 - 3\mu'_2\mu'_1 + 2(\mu'_1)^3 = -86$$

25. Find the r-th moment about origin of the R.V X with p.d.f  $f(x) =$

$$\begin{cases} Ce^{-ax}, x \geq 0 \\ 0, \text{ else where} \end{cases}$$

**Answer:**

$$\int_0^{\infty} C e^{-ax} dx = 1 \Rightarrow C = a$$

$$\mu'_r = \int_0^{\infty} x^r f(x) dx = a \int_0^{\infty} x^{(r+1)-1} e^{-ax} dx = \frac{\sqrt{(r+1)}}{a^r} = \frac{r!}{a^r}$$

26. A C.R.V X has the p.d.f  $f(x)=kx^2e^{-x}, x > 0$ . Find the r-th moment about the origin.

**Answer:**

$$\int_0^{\infty} kx^2 e^{-x} dx = 1 \Rightarrow k = \frac{1}{2}$$

$$\mu'_1 = E[X^r] = \frac{1}{2} \int_0^{\infty} x^{r+2} e^{-x} dx = \frac{1}{2} \sqrt{(r+3)} = \frac{(r+2)!}{2}$$

27.If X and Y are independent R,V's and  $Z = X+Y$ , prove that  $M_x(t)M_y(t)$ .

**Answer:**

$$\begin{aligned} M_z(t) &= E[e^{tz}] = E[e^{t(X+Y)}] = E[e^{tx}]E[e^{ty}] \\ &= M_x(t)M_y(t). \end{aligned}$$

28.If the MGF of X is  $M_x(t)$  and if  $Y=aX+b$  show that  $M_y(t) = e^{bt}M_x(at)$ .

**Answer:**

$$M_y(t) = E[e^{ty}] = E[e^{bt}e^{axt}] = e^{bt}E[e^{(at)X}] = e^{bt}M_x(at).$$

29.If a R.V X has the MGF  $M(t)=\frac{3}{3-t}$ , obtain the mean and variance of X.

**Answer:**

$$M(t) = \frac{3}{3[1-\frac{t}{3}]} = 1 + \frac{t}{3} + \frac{t^2}{9} + \dots$$

$$E(x) = \text{Co-efficient of } \frac{t}{1!} \text{ in } (1) = \frac{1}{3}$$

$$E(X^2) = \text{co-efficient of } \frac{t^2}{2!} \text{ in } (1) = \frac{1}{9}$$

$$\therefore \text{Mean} = \frac{1}{3} \text{ and } V(X) = E(X^2) - [E(X)]^2 = \frac{1}{9}$$

30. If the r-th moment of a C.R.V X about the origin is r!, find the M.G. F of X.

**Answer:**

$$\begin{aligned} M_x(t) &= \sum_{r=0}^{\infty} E[X^r] \cdot \frac{t^r}{r!} = \sum_{r=0}^{\infty} t^r \\ &= 1 + t + t^2 + \dots = (1 - t)^{-1} = \frac{1}{1 - t} \end{aligned}$$

31. If the MGF of a R.V. X is  $\frac{2}{2-t}$ , Find the standard deviation of x.

**Answer:**

$$M_x(t) = \frac{2}{2-t} = (1 - \frac{t}{2})^{-1} = 1 + \frac{t}{2} + \frac{t^2}{4} + \dots$$

$$E(X) = \frac{1}{2}; E(x^2) = \frac{1}{2}; V(X) = \frac{1}{4} \Rightarrow S.D \text{ of } X = \frac{1}{2}$$

32. Find the M.G.F of the R.V X having p.d.f  $f(x) = \begin{cases} \frac{1}{3}, & -1 < x < 2 \\ 0, & \text{else where} \end{cases}$

**Answer:**

$$M_x(t) = \int_{-1}^2 \frac{1}{3} e^{tx} dx = \frac{1}{3t} [e^{2t} - e^{-t}] \text{ for } t \neq 0$$

$$\text{When } t=0, M_x(t) = \int_{-1}^2 \frac{1}{3} dx = 1$$



$$\therefore M_x(t) = \begin{cases} \frac{e^{2t} - e^{-t}}{3t}, & t \neq 0 \\ 1, & t = 0 \end{cases}$$

33. Find the MGF of a R.V X whose moments are given by  $\mu'_r = (r + 1)!$

**Answer:**

$$\begin{aligned} M_x(t) &= \sum_{r=0}^{\infty} E[X^r] \cdot \frac{t^r}{r!} = \sum_{r=0}^{\infty} (r + 1)t^r \\ &= 1 + 2t + 3t^2 + \dots = (1 - t)^{-2} \\ \therefore M_x(t) &= \frac{1}{(1 - t)^2} \end{aligned}$$

34. Give an example to show that if p.d.f exists but M.G.F. does not exist.

**Answer:**

$$P(x) = \begin{cases} \frac{6}{\pi^2 x^2}, & x = 1, 2, \dots \\ 0, & \text{otherwise} \end{cases}$$

$$\sum P(x) = \frac{6}{\pi^2} \Rightarrow \sum_{x=1}^{\infty} \frac{1}{x^2} = \frac{6}{\pi^2} \left[ \frac{\pi^2}{6} \right] = 1$$

$\therefore P(x)$  is a p.d.f.

But  $M_x(t) = \frac{6}{\pi^2} \sum \frac{e^{tx}}{x^2}$ , which is a divergent series

$\therefore M_x(t)$  doesn't exist.

35. The moment generating function of a random variable X is given by  $M_x(t) =$

$\frac{1}{3}e^t + \frac{4}{15}e^{3t} + \frac{2}{15}e^{4t} + \frac{4}{15}e^{5t}$ . Find the probability density function of X.

**Answer:**

X	1	2	3	4
P(X)	1/3	4/15	2/15	4/15

36. Let  $M_x(t) = \frac{1}{(1-t)}$ ,  $t < 1$  be the M.G.F of a R.V X. Find the MGF of the RV

$$Y=2X+1.$$

**Answer:**

$$\text{If } Y = aX+b, M_y(t) = e^{bt}M_x(at) \therefore M_y(t) = \frac{e^t}{1-2t}.$$

37. Suppose the MGF of a RV X is of the form  $M_x(t) = (0.4e^t + 0.6)^8$ . What is the MGF of the random variable  $Y=3X+2$ .

**Answer:**

$$M_y(t) = e^{2t}M_x(3t) = e^{2t}[(0.4)e^{3t} + 0.6]^8$$

38. The moment generating function of a RV X is  $\left[\frac{1}{5} + \frac{4e^t}{5}\right]^{15}$ . Find the MGF of

$$Y=2X + 3.$$

**Answer:**

$$\text{If } Y = 2X + 3, \text{ then } M_y(t) = e^{3t}M_x(2t).$$

$$\therefore M_y(t) = e^{3t} \left[ \frac{1}{5} + \frac{4e^t}{5} \right]^{15}$$

39. If a random variable takes the values -1, 0 and 1 with equal probabilities, find the MGF of X.

**Answer:**

$$M_x(t) = \sum e^{tx} P(x) = \frac{1}{3}e^{-1} + \frac{1}{3} + \frac{1}{3}e^1 = \frac{1}{3}[1 + e^1 + e^{-1}]$$

1. Determine the binomial distribution whose mean is 9 and whose standard deviation is  $\frac{3}{2}$ .

**Answer:**

$$np = 9 \text{ and } npq = \frac{9}{4}. \quad q = \frac{npq}{np} = \frac{1}{4} \Rightarrow p = 1 - q = \frac{3}{4}$$

$$np = 9 \Rightarrow n = 9 \times \frac{4}{3} = 12$$

$$\therefore P[X=r] = {}^{12}C_r \cdot \left[\frac{3}{4}\right]^r \left[\frac{1}{4}\right]^{12-r}, r = 0, 1, 2, \dots, 12$$

2. A die is thrown 3 times. If getting a '6' is considered a success, find the probability of atleast two successes.

**Answer:**

$$P=1/6; \quad q=5/6; \quad n=3.$$

$$P[\text{atleast two successes}] = P(2) + P(3)$$

$$= {}^3C_2 \cdot \left[\frac{1}{6}\right]^2 \frac{5}{6} + {}^3C_3 \cdot \left[\frac{1}{6}\right]^3 = \frac{2}{27}$$

3. Find the MGF of binomial distribution.

**Answer:**

$$\begin{aligned} M_x(t) &= \sum_{r=0}^n {}^nC_r \cdot (pe^t)^r \cdot q^{n-r} \\ &= (q + pe^t)^n \end{aligned}$$

4. For a random variable X,  $M_x(t) = \frac{1}{81}(e^t + 2)^4$ , find  $P[X \leq 2]$ .

**Answer:**

$$M_x(t) = \left(\frac{2}{3} + \frac{1}{3}e^t\right)^4.$$

For Binomial distribution,  $M_x(t) = (q + pe^t)$

$$\therefore n=4, \quad q=2/3, \quad p=1/3$$

$$\therefore P[X \leq 2] = P(0) + P(1) + P(2)$$

$$\begin{aligned} &= \left(\frac{2}{3}\right)^4 + 4C_1 \frac{1}{3} \left(\frac{2}{3}\right)^3 + 4C_1 \left(\frac{1}{3}\right)^2 \left(\frac{2}{3}\right)^2 \\ &= \frac{1}{81} [16 + 32 + 24] = \frac{72}{81} \\ &= 0.8889 \end{aligned}$$

5. The mean and variance of a binomial variance are 4 and 4/3 respectively, find

$$P [ X \geq 1 ] .$$

**Answer:**

$$np = 4, \quad npq = \frac{4}{3} \Rightarrow q = \frac{1}{3} \text{ and } p = \frac{2}{3} \quad \therefore n = 4 \times \frac{3}{2} = 6.$$

$$P[ X \geq 1 ] = 1 - P[ X < 1 ] = 1 - P[ X = 0 ]$$

$$= 1 - \left(\frac{1}{3}\right)^6 = 0.9986$$

6. For a binomial distribution, mean is 6 and standard deviation is  $\sqrt{2}$ . Find the first two terms of the distribution.

**Answer:**

$$np = 6, \quad npq = 2; \quad q = \frac{2}{3} \Rightarrow q = \frac{1}{3} \quad \therefore p = \frac{2}{3}. \text{ Here } n = 9.$$

$$\text{The first two terms are } \left(\frac{1}{3}\right)^9, 9C_1 \left(\frac{2}{3}\right) \left(\frac{1}{3}\right)^8$$

7. A certain rare blood can be found in only 0.05% of people. If the population of a randomly selected group is 3000, what is the probability that atleast 2 people in the group have this rare blood type ?

**Answer:**

$$P=0.05\% \quad \Rightarrow p=0.0005; n = 3000; \lambda = np$$

$$\Rightarrow \lambda = 3000 \times \frac{5}{10000} = 1.5$$

$$P[X \geq 2] = 1 - P(X < 2) = 1 - P(X=1) \\ = 1 - e^{-1.5} \left( 1 + \frac{1.5}{1!} \right) = 0.4422$$

8. It is known that 5% of the books bound at a certain bindery have defective bindings. Find the probability that 2 of 100 books bound by this bindery will have defective bindings.

**Answer:**

$$\lambda = np \Rightarrow \lambda = 100 \times 5/100 = 5$$

$$\therefore P[X=2] = \frac{5^2 e^{-5}}{2!} = 0.084$$

9. If X is a poissonvariate such that  $P(X=2) = 9P(X=4) + 90P(X=6)$ , find the variance .

**Answer:**

$$\frac{e^{-\lambda} \lambda^2}{2!} = \frac{9e^{-\lambda} \lambda^4}{4!} + \frac{90e^{-\lambda} \lambda^6}{6!} \Rightarrow \lambda^4 + 3\lambda^2 - 4 = 0$$

$$\Rightarrow (\lambda^2 + 4)(\lambda^2 - 1) = 0$$

$$\therefore \lambda^2 = 1 \Rightarrow \text{variance} = \lambda = 1.$$

10. The moment generating function of a random variable X is given by  $M_x(t) = e^{3(e^t-1)}$ . Find  $P(X=1)$

**Answer:**

$$M_x(t) = e^{\lambda(e^t-1)} = e^{3(e^t-1)} \Rightarrow \lambda = 3$$

$$P(X=1) = \lambda e^{-\lambda} \Rightarrow P(X=1) = 3e^{-3}.$$

11. State the conditions under which the poisson distribution is a limiting case of the Binomial distribution.

**Answer:**

i)  $n \rightarrow \infty$

ii)  $p \rightarrow 0$

iii)  $np = \lambda$ , a constant

12. Show that the sum of 2 independent poisson variates is a poisson variate.

**Answer:**

Let  $X \sim P(\lambda_1)$  and  $Y \sim P(\lambda_2)$

Then  $M_x(t) = e^{\lambda_1(e^t-1)}$ ;  $M_y(t) = e^{\lambda_2(e^t-1)}$

$M_{x+y}(t) = M_x(t)M_y(t) = e^{(e^t-1)(\lambda_1+\lambda_2)}$

$\Rightarrow X + Y$  is also a poisson variate

13. In a book of 520 pages, 390 typo-graphical errors occur. Assuming poisson law for the number of errors per page, find the probability that a random sample of 5 pages will contain no error.

**Answer:**

$$\lambda = \frac{390}{520} = 0.75$$

$$P(X=x) = \frac{e^{-\lambda} \lambda^x}{x!} = \frac{e^{-0.75} (0.75)^x}{x!}, \quad x = 0, 1, 2, \dots$$

Required probability =  $[P(X = 0)]^5 = (e^{-0.75})^5 = e^{-3.75}$

14. If  $X$  is a poisson variate such that  $P(X=2) = \frac{2}{3} P(X=1)$  evaluate  $P(X=3)$ .

**Answer:**

$$\frac{e^{-\lambda} \lambda^2}{2!} = \frac{2 e^{-\lambda} \lambda}{3 \cdot 1!} \Rightarrow \lambda = \frac{4}{3}$$

$$\therefore P[X=3] = \frac{e^{-\lambda} \left(\frac{4}{3}\right)^3}{3!}$$

15. If for a poisson variate  $X$ ,  $E(X^2) = 6$ , What is  $E(X)$ ?

**Answer:**

$$\lambda^2 + \lambda = 6 \Rightarrow \lambda^2 + \lambda - 6$$

## PROBABILITY AND RANDOM PROCESSES

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$$= 0 \Rightarrow (\lambda + 3)(\lambda - 2) = 0 \Rightarrow \lambda = 2, -3$$

But  $\lambda > 0$ ,  $\lambda = 2$ .  $E(X) = \lambda = 2$

16. If  $X$  is a poisson variate with mean  $\lambda$ , show that  $E(X^2) = \lambda E(X + 1)$ .

**Answer:**

$$E(X^2) = \lambda^2 + \lambda$$

$$E(X+1) = E(X)+1 = \lambda + 1$$

$$\therefore E(X^2) = \lambda (\lambda + 1) = \lambda E(X + 1)$$

17. The time (in hours) required to repair a machine is exponentially distributed

with parameter  $\lambda = \frac{1}{2}$ . What is the probability that a repair takes atleast 10

hours given that its duration exceeds 9 hours ?

**Answer:**

Let  $X$  be the R.V which represents the time to repair the machine.

$$P[X \geq 10/x \geq 9] = P(X \geq 1) \text{ (by memory less property)}$$

$$= \int_1^{\infty} \frac{1}{2} e^{-\frac{x}{2}} dx = 0.6065$$

18. The time (in hours) required to repair a machine is exponentially distributed

with parameter  $\lambda = \frac{1}{3}$ . What is the probability that the repair time exceeds 3

hours ?

**Answer:**

$X$ - represent the time to repair the machine

$$\text{P.d.f of } X, f(x) = \frac{1}{3} e^{-\frac{x}{3}}, x > 0$$

$$P(x > 3) = \int_3^{\infty} \frac{1}{3} e^{-\frac{x}{3}} dx = e^{-1} = 0.3679$$

19. Find the MGF of an exponential distribution with parameter  $\lambda$ .

**Answer:**

$$M_x(t) = \lambda \int_0^{\infty} e^{tx} e^{-\lambda x} dx = \lambda \int_0^{\infty} e^{-(\lambda - t)x} dx$$

$$= \frac{\lambda}{\lambda - t} = \left(1 - \frac{t}{\lambda}\right)^{-1}$$

20. Mention any four properties of normal distribution ?

**Answer:**

- (1) The curve is bell shaped
- (2) Mean, Median, Mode coincide.
- (3) All odd central moments vanish
- (4) X-axis is an asymptote to the normal curve

21. If X is normal variate with mean 30 and S.D 5, find  $P[26 < X < 40]$

**Answer:**

$$\begin{aligned} P [26 < X < 40] &= P [-0.8 \leq Z \leq 2] \text{ where } Z = \frac{X-30}{5} \\ &= P [0 \leq Z \leq 0.8] + P[0 \leq Z \leq 2] \\ &= 0.2881 + 0.4772 = 0.7653 \end{aligned}$$

22. If X is a normal variate with mean 30 and s.d 5, find  $P [|X - 30| \leq 5]$ .

**Answer:**

$$\begin{aligned} P [|X - 30| \leq 5] &= P [25 \leq X \leq 35] = P [-1 \leq Z \leq 1] \\ &= 2P (2 \leq Z \leq 1) = 2(0.3413) = 0.6826 \end{aligned}$$

23. X is normally distributed R.V with mean 12 and SD 4. Find  $P [ X \leq 20 ]$ .

**Answer:**

$$\begin{aligned} P [ X \leq 20 ] &= P [ Z \leq 2 ] \text{ where } Z = \frac{X-12}{4} \\ &= P [ -\infty \leq Z \leq 0 ] + P [ 0 \leq Z \leq 2 ] \end{aligned}$$



$$= 0.5 + 0.4772 = 0.9772$$

24. For a certain normal distribution, the first moment about 10 is 40 and the fourth moment about 50 is 48. Find the mean and s.d of the distribution.

**Answer:**

$$\text{Mean } A + \mu'_1 \Rightarrow \text{Mean} = 10 + 40 = 50$$

$$\mu'_4(\text{ about the point } X = 50) = 48 \Rightarrow \mu_4 = 48$$

$$\text{Since mean is } 50, 3\sigma^4 = 48$$

$$\sigma = 2.$$

25. If X is normally distributed with mean 8 and s.d 4, find  $P(10 \leq X \leq 15)$ .

**Answer:**

$$\begin{aligned} P(10 \leq X \leq 15) &= P[0.5 \leq X \leq 1.75] \\ &= P[0.5 \leq X \leq 1.75] - P[0 \leq X \leq 0.5] \\ &= 0.2684 \end{aligned}$$

26. X is a normal variate with mean 1 and variance 4, Y is another normal variate independent of X with mean 2 and variance 3, what is the distribution of

$X + 2Y$  ?

**Answer:**

$$\begin{aligned} E[X + 2Y] &= E(X) + 2E(Y) = 1 + 4 = 5 \\ V[X + 2Y] &= V(X) + 4V(Y) = 4 + 4(3) = 16 \\ X + 2Y &\sim N(5, 16) \text{ by additive property.} \end{aligned}$$

**UNIT 2**

Two marks

1. The bivariate random variable X and Y has the pdf  $f(x,y)=\{$

$$f(x,y)=\begin{cases} kx^2(8-y), & x < y < 2x \\ 0 & 0 \leq x < 2 \end{cases} \quad \text{find } k.$$

Ans:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y) dy dx = 1$$

$$\int_0^2 \int_x^{2x} kx^2(8-y) dy dx = 1 \quad k \int_0^2 x^2 \left( 8y - \frac{y^2}{2} \right) \Big|_x^{2x} dx = 1$$

$$k \int_0^2 x^2 \left( 16x - \frac{4x^2}{2} - 8x + \frac{x^2}{2} \right) dx = 1 \quad k \int_0^2 \left( 16x^3 - 2x^4 - 8x^3 + \frac{x^4}{2} \right) dx = 1$$

$$k \int_0^2 \left( 8x^3 - \frac{3x^4}{2} \right) dx = 1 \quad k \left[ \frac{8x^4}{4} - \frac{3x^5}{10} \right]_0^2 = 1$$

$$k \left[ 32 - \frac{48}{5} \right] = 1 \quad k \left[ \frac{112}{5} \right] = 1$$

$$k \left[ \frac{112}{5} \right] = 1$$

$$k = \frac{5}{112}$$

2. The joint pdf of random variable x and y is given by  $f(x,y) = kxye^{-(x^2+y^2)}$ ,  $x > 0, y > 0$   
find the value of k.

Ans:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y) dx dy = 1$$

$$\int_0^{\infty} \int_0^{\infty} kxye^{-(x^2+y^2)} dy dx = 1$$

## PROBABILITY AND RANDOM PROCESSES

---

$$k \int_0^{\infty} y e^{-y^2} dy \int_0^{\infty} x e^{-x^2} dx = 1 \quad \left[ \int_0^{\infty} x e^{-x^2} dx = \frac{1}{2} \right]$$

$$k \frac{1}{2} \cdot \frac{1}{2} = 1, \quad k = 4$$

3. If X and Y have joint pdf  $f(x,y) = \begin{cases} x+y, & 0 < x < 1, 0 < y < 1 \\ 0, & \text{otherwise} \end{cases}$ . check whether X and Y are independent.

Ans:

The marginal density of X is

$$f(x) = \int_{-\infty}^{\infty} f(x,y) dy \quad f(x) = \int_0^1 (x+y) dy$$

$$f(x) = \left[ xy + \frac{y^2}{2} \right]_0^1 \quad f(x) = x + \frac{1}{2}$$

$$f(y) = \int_{-\infty}^{\infty} f(x,y) dx \quad f(y) = \int_0^1 (x+y) dx$$

$$f(y) = \left[ \frac{x^2}{2} + xy \right]_0^1 \quad f(y) = \frac{1}{2} + y$$

$$f(x) \cdot f(y) = \left( x + \frac{1}{2} \right) \left( y + \frac{1}{2} \right) \quad f(x) \cdot f(y) \neq f(x,y)$$

4. Let X and Y have j.d.f  $f(x,y)=2, 0 < x < y < 1$ . Find m.d.f

Ans:

Marginal density of X is given by

$$f(x) = \int_{-\infty}^{\infty} f(x,y) dy = \int_x^1 2 dy$$

$$= 2[y]_x^1$$

$$= 2(1-x), 0 < x < 1.$$

Marginal density function of Y is given by

$$f(y) = \int_{-\infty}^{\infty} f(x,y)dy = \int_0^y 2dx = 2[x]_0^y = 2y, 0 < y < 1.$$

5. The j.d.f of the random variables X and Y is given by

$$f(x,y) = \begin{cases} 8xy, & 0 < x < 1, 0 < y < x \\ 0, & \text{otherwise} \end{cases} \text{.find } f_x(x).$$

Ans:

$$\begin{aligned} f_x(x) &= \int_{-\infty}^{\infty} f(x,y)dy \\ &= \int_0^x 8xydy = 8x \left( \frac{y^2}{2} \right)_0^x \\ &= 8x \left( \frac{x^2}{2} \right) \end{aligned}$$

$$f_x(x) = 4x^3, 0 < x < 1$$

6. Given  $f(x,y) = \begin{cases} cx(x-y), & 0 < x < 2, -x < y < x \\ 0, & \text{otherwise} \end{cases}$ , find c.

Ans:

$$\begin{aligned} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y)dydx &= 1 \\ \int_0^2 \int_{-x}^x cx(x-y)dydx &= 1 \\ c \int_0^2 \left[ x^2y - x \cdot \frac{y^2}{2} \right]_{-x}^x dx &= 1 \\ c \int_0^2 \left[ x^3 - \frac{x^2}{2} + x^3 + \frac{x^3}{2} \right] dx &= 1 \end{aligned}$$

$$c \int_0^2 2x^3 dx = 1$$

$$2c \left[ \frac{x^4}{4} \right]_0^2 = 1 \qquad 2c \left[ \frac{16}{4} \right] = 1 \qquad c = \frac{1}{8}$$

6. The joint p.d.f of a bivariate random variable (X,Y) is given by

$$f(x,y) = \begin{cases} kxy, & 0 < x < 1, 0 < y < 1 \\ 0, & \text{otherwise} \end{cases}, \text{ find K.}$$

Ans:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y) dx dy = 1$$

$$\int_0^1 \int_0^1 kxy dx dy = 1 \qquad k \int_0^1 \left[ \frac{x^2}{2} y \right]_0^1 dy = 1$$

$$k \int_0^1 \frac{y}{2} dy = 1 \qquad k \left[ \frac{y^2}{4} \right]_0^1 = 1$$

$$k = 4$$

7. If the joint pdf of (x,y) is  $f(x,y) = \frac{1}{4}, 0 < x, y < 1$ , find  $p(x+y \leq 1)$ .

Ans:

$$p(x+y \leq 1) = p(x \leq 1-y)$$

$$= \int_0^{1-y} \int_0^{1-y} f(x,y) dx dy$$

$$= \int_0^{1-y} \int_0^{1-y} \frac{1}{4} dx dy = \frac{1}{4} \int_0^{1-y} [x]_0^{1-y} dy$$

$$= \frac{1}{4} \int_0^1 [(1-y)] dy \qquad = \frac{1}{4} \left[ y - \frac{y^2}{2} \right]_0^1$$

## PROBABILITY AND RANDOM PROCESSES

---

$$= \frac{1}{4} \left[ 1 - \frac{1}{2} \right] = \frac{1}{4} \cdot \frac{1}{2} = \frac{1}{8}$$

8. Two random variables X and Y have joint pdf  $f(x,y) = \begin{cases} \frac{xy}{96}, & 0 < x < 4, 1 < y < 5 \\ 0, & \text{otherwise} \end{cases}$ , find

E(x).

Ans:

$$\begin{aligned} E(x) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xf(x,y) dx dy \\ &= \int_1^5 \int_0^4 x \left( \frac{xy}{96} \right) dx dy = \frac{1}{96} \int_1^5 \left[ y \cdot \frac{x^3}{3} \right]_0^4 dy \\ &= \frac{1}{96} \int_1^5 \frac{64}{3} y dy = \frac{64}{288} \left[ \frac{y^2}{2} \right]_1^5 \\ &= \frac{2}{9} \left[ \frac{25}{2} - \frac{1}{2} \right] = \frac{1}{9} (24) \\ &= \frac{8}{3}. \end{aligned}$$

9. Let X be a Random variable with pdf  $f(x) = \frac{1}{2}, -1 \leq x \leq 1$ , and let  $Y = X^2$ , find E(Y).

Ans:

$$Y = x^2$$

$$\begin{aligned} E(Y) &= E(x^2) \\ \int_{-\infty}^{\infty} x^2 f(x) dx &= \int_{-\infty}^{\infty} x^2 \left( \frac{1}{2} \right) dx = \frac{1}{2} \left( \frac{x^3}{3} \right)_{-1}^1 = \frac{1}{6} (2) = \frac{1}{3}. \end{aligned}$$

10. If the joint pdf of (x,y) is given by  $f(x,y) = x+y, 0 \leq x, y \leq 1$ . Find  $E(XY)$ .

Ans:

$$E(XY) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xyf(x,y) dx dy$$

## PROBABILITY AND RANDOM PROCESSES

---

$$\begin{aligned} &= \int_0^1 \int_0^1 xy(x+y) dx dy \\ &= \int_0^1 \int_0^1 (x^2 y + xy^2) dx dy \\ &= \int_0^1 \left( \frac{x^3}{3} y + \frac{x^2}{2} y^2 \right)_0^1 dy \\ &= \int_0^1 \left( \frac{y}{3} + \frac{y^2}{2} \right) dy \\ &= \left( \frac{y^2}{6} + \frac{y^3}{6} \right)_0^1 = \frac{2}{6} = \frac{1}{3}. \end{aligned}$$

11. Find the acute angle between the two lines of regression

Ans:

The acute angle between the two lines of regression is  $\tan \theta = \frac{1-r^2}{r} \left( \frac{\sigma_x \sigma_y}{\sigma_x^2 + \sigma_y^2} \right)$ .

12. State the equation of the two regression lines. What is the formula for correlation coefficient

Ans:

X on Y is  $(x - \bar{x}) = b_{xy}(y - \bar{y})$  and Y on X is  $(y - \bar{y}) = b_{yx}(x - \bar{x})$ .

Correlation coefficient  $r = \sqrt{b_{xy} \cdot b_{yx}}$ .

13. If X and Y are independent random variables with variance 2 and 3. Find the variance of  $3X+4Y$ .

Ans:

$$\begin{aligned} \text{Var}(x) &= 2, \text{Var}(y) = 3 \\ \text{Var}(3X + 4Y) &= 3^2 \text{Var}(X) + 4^2 \text{Var}(Y) \\ &= 9\text{Var}(X) + 16\text{Var}(Y) \\ &= 9*2 + 16*3 \\ &= 66 \end{aligned}$$

14. The joint pdf of (X,Y) is given by  $e^{-(x+y)}$ ,  $0 < x, y < \infty$ . Are X and Y independent?

Ans :

$$f(x) = \int_{-\infty}^{\infty} f(x, y) dy$$

## PROBABILITY AND RANDOM PROCESSES

---

$$\begin{aligned} &= \int_0^{\infty} e^{-x} e^{-y} dy \\ &= e^{-x} \left( -e^{-y} \right)_0^{\infty} = -e^{-x} (0-1) = e^{-x}. \end{aligned}$$

$$\begin{aligned} f(y) &= \int_{-\infty}^{\infty} f(x, y) dx \\ &= \int_0^{\infty} e^{-x} e^{-y} dx \\ &= e^{-y} \left( -e^{-x} \right)_0^{\infty} = -e^{-y} (0-1) = e^{-y}. \end{aligned}$$

$$f(x)f(y) = e^{-x} e^{-y} = e^{-(x+y)} = f(x, y) \quad \text{Therefore, X and Y are independent.}$$

15. The two lines of regression are  $8x - 10y + 66 = 0$ ,  $40x - 18y - 214 = 0$ . Find the mean value of X and Y.

Ans:

$$8x - 10y = -66 \quad (1)$$

$$40x - 18y = 214 \quad (2)$$

Solving (1) and (2), we get  $x = 104$ ,  $y = 17$

Mean of X = 13

Mean of Y = 17.

16. The two regression lines are  $x = \frac{9}{20}y + \frac{107}{20}$ ,  $y = \frac{4}{5}x + \frac{33}{5}$ . Find correlation coefficient?

Ans:

$$r = \sqrt{b_{xy} \cdot b_{yx}}$$

$$\text{Here, } b_{xy} = \frac{9}{20}, b_{yx} = \frac{4}{5}$$

$$r = \sqrt{\frac{9}{20} \times \frac{4}{5}} = 0.6$$



## UNIT 3

Two marks

**1. Define Random processes and give an example of a random process.**

A Random process is a collection of R.V  $\{X(s,t)\}$  that are functions of a real variable namely time  $t$  where  $s \in S$  and  $t \in T$

**Example:**

$X(t) = A \cos(\omega t + \theta)$  where  $\theta$  is uniformly distributed in  $(0, 2\pi)$  where  $A$  and  $\omega$  are constants.

**2. State the four classifications of Random processes.**

The Random processes is classified into four types

(i) Discrete random sequence

If both  $T$  and  $S$  are discrete then Random processes is called a discrete Random sequence.

(ii) Discrete random processes

If  $T$  is continuous and  $S$  is discrete then Random processes is called a Discrete Random processes.

(iii) Continuous random sequence

If  $T$  is discrete and  $S$  is continuous then Random processes is called a Continuous Random sequence.

(iv) Continuous random processes

If  $T$  &  $S$  are continuous then Random processes is called a continuous Random processes.

**3. Define stationary Random processes.**

If certain probability distributions or averages do not depend on  $t$ , then the random process  $\{X(t)\}$  is called stationary.

**4. Define first order stationary Random processes.**

A random processes  $\{X(t)\}$  is said to be a first order SSS process if  $f(x_1, t_1 + \delta) = f(x_1, t_1)$  (i.e.) the first order density of a stationary process  $\{X(t)\}$  is independent of time  $t$

**5. Define second order stationary Random processes**

ARP  $\{X(t)\}$  is said to be second order SSS if  $f(x_1, x_2, t_1, t_2) = f(x_1, x_2, t_1 + h, t_2 + h)$  where  $f(x_1, x_2, t_1, t_2)$  is the joint PDF of  $\{X(t_1), X(t_2)\}$ .

## 6. Define strict sense stationary Random processes

Sol: A RP  $\{X(t)\}$  is called a SSS process if the joint distribution

$X(t_1)X(t_2)X(t_3)\dots\dots X(t_n)$  is the same as that of

$X(t_1+h)X(t_2+h)X(t_3+h)\dots\dots X(t_n+h)$  for all  $t_1, t_2, t_3, \dots, t_n$  and  $h > 0$  and for  $n \geq 1$ .

## 7. Define wide sense stationary Random processes

A RP  $\{X(t)\}$  is called WSS if  $E\{X(t)\}$  is constant and  $E[X(t)X(t+\tau)] = R_{xx}(\tau)$

(i.e.) ACF is a function of  $\tau$  only.

## 8. Define jointly strict sense stationary Random processes

Sol: Two real valued Random Processes  $\{X(t)\}$  and  $\{Y(t)\}$  are said to be jointly stationary in the strict sense if the joint distribution of the  $\{X(t)\}$  and  $\{Y(t)\}$  are invariant under translation of time.

## 9. Define jointly wide sense stationary Random processes

Sol: Two real valued Random Processes  $\{X(t)\}$  and  $\{Y(t)\}$  are said to be jointly stationary in the wide sense if each process is individually a WSS process and  $R_{xy}(t_1, t_2)$  is a function of  $t_1, t_2$  only.

## 10. Define Evolutionary Random processes and give an example.

Sol: A Random processes that is not stationary in any sense is called an Evolutionary process. Example: Poisson process.

## 11. When is a random process said to be ergodic? Give an example

Answer: A R.P  $\{X(t)\}$  is ergodic if its ensemble averages equal to appropriate time averages. Example:  $X(t) = A \cos(\omega t + \theta)$  where  $\theta$  is uniformly distributed in  $(0, 2\pi)$  is mean ergodic.

## 12. Define Markov Process.

Sol: If for  $t_1 < t_2 < t_3 < t_4 \dots \dots \dots < t_n < t$  then

$$P(X(t) \leq x / X(t_1) = x_1, X(t_2) = x_2, \dots \dots \dots X(t_n) = x_n) = P(X(t) \leq x / X(t_n) = x_n)$$

Then the process  $\{X(t)\}$  is called a Markov process.

## 13. Define Markov chain.

Sol: A Discrete parameter Markov process is called Markov chain.

## 14. Define one step transition probability.

Sol: The one step probability  $P[X_n = a_j / X_{n-1} = a_i]$  is called the one step probability from the state  $a_i$  to  $a_j$  at the  $n^{\text{th}}$  step and is denoted by  $P_{ij}(n-1, n)$

**15. State the postulates of a Poisson process.**

Let  $\{X(t)\}$  = number of times an event A say, occurred up to time 't' so that the sequence  $\{X(t)\}$ ,  $t \geq 0$  forms a Poisson process with parameter  $\lambda$ .

- (i)  $P[1 \text{ occurrence in } (t, t + \Delta t)] = \lambda \Delta t$
- (ii)  $P[0 \text{ occurrence in } (t, t + \Delta t)] = 1 - \lambda \Delta t$
- (iii)  $P[2 \text{ or more occurrence in } (t, t + \Delta t)] = 0$
- (iv)  $X(t)$  is independent of the number of occurrences of the event in any interval prior and after the interval  $(0, t)$ .
- (v) The probability that the event occurs a specified number of times in  $(t_0, t_0 + t)$  depends only on t, but not on  $t_0$ .

**16. State any two properties of Poisson process**

Sol: (i) The Poisson process is a Markov process

(ii) Sum of two independent Poisson processes is a Poisson process

(iii) The difference of two independent Poisson processes is not a Poisson process.

**17. If the customers arrived at a counter in accordance with a Poisson process with a mean rate of 2 per minute, find the probability that the interval between two consecutive arrivals is more than one minute.**

Sol: The interval T between 2 consecutive arrivals follows an exponential distribution with

$$\text{parameter } \lambda = 2, P(T > 1) = \int_1^{\infty} 2e^{-2t} dt = e^{-2} = 0.135.$$

**18. A bank receives on an average  $\lambda = 6$  bad checks per day, what are the probabilities that it will receive (i) 4 bad checks on any given day (ii) 10 bad checks over any 2 consecutive days.**

$$\text{Sol: } P(X(t) = n) = \frac{e^{-\lambda t} (\lambda t)^n}{n!} = \frac{e^{-6t} (6t)^n}{n!}, n = 0, 1, 2, \dots$$

$$(i) P(X(1) = 4) = \frac{e^{-6} (6)^4}{4!} = 0.1338$$

$$(ii) P(X(2) = 10) = \frac{e^{-12} (12)^{10}}{10!} = 0.1048$$

19. Consider a Markov chain with two states and transition probability matrix

$$P = \begin{bmatrix} 3/4 & 1/4 \\ 1/2 & 1/2 \end{bmatrix}. \text{ Find the stationary probabilities of the chain.}$$

$$\text{Sol: } (\pi_1, \pi_2) \begin{bmatrix} 3/4 & 1/4 \\ 1/2 & 1/2 \end{bmatrix} = (\pi_1, \pi_2) \quad \pi_1 + \pi_2 = 1$$

$$\frac{3}{4}\pi_1 + \frac{\pi_2}{4} = \pi_1 \Rightarrow \frac{\pi_1}{4} - \frac{\pi_2}{2} = 0. \quad \therefore \pi_1 = 2\pi_2$$

$$\therefore \pi_1 = \frac{2}{3}, \pi_2 = \frac{1}{3}.$$

## UNIT 4

### Two marks

#### 1. Define the ACF.

**Answer:**

Let  $X(t_1)$  and  $X(t_2)$  be two random variables. The autocorrelation of the random process  $\{X(t)\}$  is

$$R_{XX}(t_1, t_2) = E[X(t_1)X(t_2)].$$

If  $t_1 = t_2 = t$ ,  $R_{XX}(t, t) = E[X^2(t)]$  is called as mean square value of the random process.

#### 2. State any four properties of Autocorrelation function.

**Answer:**

1.  $R_{XX}(-\tau) = R_{XX}(\tau)$

2.  $|R(\tau)| \leq R(0)$

3.  $R(\tau)$  is continuous for all  $\tau$ .

4. If  $R(\tau)$  is ACF of a stationary RP  $\{X(t)\}$  with no periodic components, then

$$\mu_X^2 = \lim_{\tau \rightarrow \infty} R(\tau).$$

#### 3. Define the cross – correlation function.

**Answer:**

Let  $\{X(t)\}$  and  $\{Y(t)\}$  be two random processes. The cross-correlation is

$$R_{XY}(\tau) = E[X(t)Y(t - \tau)].$$

#### 4. State any two properties of cross-correlation function.

**Answer:**

1.  $R_{YX}(-\tau) = R_{XY}(\tau)$

2.  $|R_{XY}(\tau)| \leq \sqrt{R_{XX}(0)R_{YY}(0)} \leq \frac{1}{2}[R_{XX}(0) + R_{YY}(0)]$

5. Given the ACF for a stationary process with no periodic component

is  $R_{XX}(\tau) = 25 + \frac{4}{1+6\tau^2}$ . find the mean and variance of the process  $\{X(t)\}$

**Answer:**

By the property of ACF

$$\mu_x^2 = \lim_{\tau \rightarrow \infty} R_{XX}(\tau) = \lim_{\tau \rightarrow \infty} 25 + \frac{4}{1+6\tau^2} = 25$$

$$\mu_x = 5$$

$$E\{X^2(t)\} = R_{XX}(0) = 25 + 4 = 29$$

$$\text{Var}\{X(t)\} = E\{X^2(t)\} - E^2\{X(t)\} = 29 - 25 = 4.$$

6. ACF:  $R_{XX}(\tau) = \frac{25\tau^2 + 36}{6.25\tau^2 + 4}$ . find mean and variance.

7. ACF:  $R_{XX}(\tau) = 25 + \frac{4}{1+6\tau^2}$ . find mean and variance.

8. Define power spectral density.

**Answer:**

If  $R_{XX}(\tau)$  is the ACF of a WSS process  $\{X(t)\}$  then the power spectral density

$S_{XX}(\omega)$  of the process  $\{X(t)\}$ , is defined by

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-i\omega\tau} d\tau \quad (\text{or}) \quad S_{XX}(f) = \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-i2\pi f\tau} d\tau$$

9. Express each of ACF and PSD of a stationary R.P in terms of the other. {or} write down wiener khinchine relation }

**Answer:**

$R_{XX}(\tau)$  and  $S_{XX}(\omega)$  are Fourier transform pairs.

$$\text{i.e., } S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-i\omega\tau} d\tau \quad \text{and} \quad R_{XX}(\tau) = \int_{-\infty}^{\infty} S_{XX}(\omega) e^{i\omega\tau} d\omega$$

10. Define cross power spectral density of two random process {X(t)} and {Y(t)}.

**Answer:**

If {X(t)} and {Y(t)} are jointly stationary random processes with cross correlation function  $R_{XY}(\tau)$ , then cross power spectral density of {X(t)} and {Y(t)} is defined by

$$S_{XY}(\omega) = \int_{-\infty}^{\infty} R_{XY}(\tau) e^{-i\omega\tau} d\tau$$

11. State any two properties of power spectral density.

**Answer:**

i)  $S(\omega) = S(-\omega)$

ii)  $S(\omega) > 0$

iii) The spectral density of a process {X(t)}, real or complex, is a real function of  $\omega$  and non-negative.

12. If  $R(\tau) = e^{-2\lambda|\tau|}$  is the ACF of a R.P {X(t)}, obtain the spectral density.

**Answer:**

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-i\omega\tau} d\tau = \int_{-\infty}^{\infty} e^{-2\lambda|\tau|} e^{-i\omega\tau} d\tau = 2 \int_0^{\infty} e^{-2\lambda\tau} \cos \omega\tau d\tau = \frac{4\lambda}{4\lambda^2 + \omega^2}$$

13. State any four properties of cross power density spectrum.

**Answer:**

i)  $S_{XY}(\omega) = S_{YX}(-\omega) = S_{YX}^*(\omega)$

ii)  $\text{Re}[S_{XY}(\omega)]$  and  $\text{Re}[S_{YX}(\omega)]$  are even function of  $\omega$

iii)  $\text{Im}[S_{XY}(\omega)]$  and  $\text{Im}[S_{YX}(\omega)]$  are odd function of  $\omega$

iv)  $S_{XY}(\omega) = 0$  and  $S_{YX}(\omega) = 0$  if X(t) and Y(t) are orthogonal.



Unit 5

Two marks

**14. Define a system and Define the linear system.**

**Answer:**

A system is a functional relationship between the input  $X(t)$  and the output  $Y(t)$ . i.e.,  $Y(t) = f[X(t)]$ ,  $-\infty < t < \infty$ .  
A System is a functional relationship between the input  $X(t)$  and the output  $Y(t)$ .  
If  $f[a_1X_1(t) + a_2X_2(t)] = a_1 f[X_1(t)] + a_2 f[X_2(t)]$ , then  $f$  is called a linear system.

**15. Define time invariant system.**

**Answer:**

If  $Y(t+h) = f[X(t+h)]$  where  $Y(t) = f[X(t)]$ , then  $f$  is called the time invariant system.

**16. Check whether the following system is linear . $y(t)=t x(t)$**

**Answer:**

Consider two input functions  $x_1(t)$  and  $x_2(t)$ . The corresponding outputs are  
 $y_1(t)=t x_1(t)$  and  $y_2(t)=t x_2(t)$   
Consider  $y_3(t)$  as the linear combinations of the two inputs.  
 $y_3(t)= t[a_1 x_1(t)+a_2 x_2(t)]= a_1t x_1(t)+a_2 t x_2(t)$  .....(1)  
consider the linear combinations of the two outputs.  
 $a_1y(t)+a_2 y_2(t)= a_1t x_1(t)+a_2 t x_2(t)$  .....(2)  
From (1)and(2), (1)=(2)  
The system  $y(t)=t x(t)$  is linear.

17.

**Check whether the following system is linear . $y(t)= x^2(t)$**

**Answer:**

Consider two input functions  $x_1(t)$  and  $x_2(t)$ . The corresponding outputs are  
 $y_1(t)=x_1^2(t)$  and  $y_2(t)=x_2^2(t)$   
Consider  $y_3(t)$  as the linear combinations of the two inputs.  
 $y_3(t)= [a_1 x_1(t)+a_2 x_2(t)]^2= a_1^2 x_1^2(t)+a_2^2 x_2^2(t)+2 a_1x_1(t)a_2x_2(t)$  .....(1)  
consider the linear combinations of the two outputs.  
 $a_1y(t)+a_2 y_2(t)= a_1 x_1^2(t)+a_2 x_2^2(t)$  .....(2)  
From (1) and (2), (1)  $\neq$  (2)  
The system  $y(t)=x^2(t)$  is not linear.



**18. Define the Linear Time Invariant System.**

**Answer:**

A linear system is said to be also time-invariant if the form of its impulse response  $h(t, u)$  does not depend on the time that the impulse is applied.

For linear time invariant system,  $h(t, u) = h(t - u)$

If a system is such that its Input  $X(t)$  and its Output  $Y(t)$  are related by a Convolution integral,

$$\text{i.e., if } Y(t) = \int_{-\infty}^{\infty} h(u) X(t - u) du, \text{ then the system is a}$$

linear time-invariant system.

**19. Find the ACF of the random process  $\{X(t)\}$ , if its power spectral density is given by**

$$S(\omega) = \begin{cases} 1 + \omega^2, & \text{for } |\omega| \leq 1 \\ 0 & , \text{for } |\omega| > 1 \end{cases}$$

**Solution:**

$$\begin{aligned} R(\tau) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega) e^{i\omega\tau} d\omega = \frac{1}{2\pi} \int_{-1}^1 \{1 + \omega^2\} e^{i\omega\tau} d\omega = \frac{1}{2\pi} \int_{-1}^1 \{e^{i\omega\tau} + \omega^2 e^{i\omega\tau}\} d\omega = \frac{1}{2\pi} \left\{ \left[ \frac{e^{i\omega\tau}}{i\tau} \right]_{-1}^1 + \int_{-1}^1 \omega^2 \cos \omega\tau d\omega \right\} \\ &= \frac{1}{2\pi} \left\{ \left[ \frac{e^{i\omega\tau}}{i\tau} \right]_{-1}^1 + 2 \int_0^1 \omega^2 \cos \omega\tau d\omega \right\} = \frac{1}{2\pi} \left[ \frac{e^{i\tau} - e^{-i\tau}}{i\tau} \right] + 2 \left[ \omega^2 \frac{\sin \omega\tau}{\tau} + \frac{2\omega \cos \omega\tau}{\tau^2} - \frac{2 \sin \omega\tau}{\tau^3} \right]_0^1 \\ &= \frac{1}{2\pi} \left[ \frac{2 \sin \tau}{\tau} + \frac{2 \sin \tau}{\tau} + \frac{4 \cos \tau}{\tau^2} - \frac{4 \sin \tau}{\tau^3} \right] = \frac{1}{2\pi} \left[ \frac{2\tau^2 \sin \tau + 2\tau^2 \sin \tau + \tau 4 \cos \tau - 4 \sin \tau}{\tau^3} \right] \\ &= \frac{2\{\tau^2 \sin \tau + \tau \cos \tau - \sin \tau\}}{\pi\tau^3} \end{aligned}$$

**20. Define Average power.**

$$\text{Average power} = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega) d\omega = R(0)$$

## PROBABILITY AND RANDOM PROCESSES

21. A system has an impulse response  $h(t) = e^{-\beta t}U(t)$ , find the system transfer function.

Solution:

The unit step function  $U(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$

$$h(t) = \begin{cases} 0, & t < 0 \\ e^{-\beta t}, & t \geq 0 \end{cases}$$
$$\therefore H(w) = \int_{-\infty}^{\infty} h(t) e^{-iwt} dt$$
$$= \int_{-\infty}^{\infty} e^{-\beta t} e^{-iwt} dt$$
$$= \int_0^{\infty} e^{-(\beta+iw)t} dt$$
$$= \left[ \frac{e^{-(\beta+iw)t}}{-(\beta+iw)} \right]_0^{\infty}$$
$$= -\frac{1}{\beta+iw} \left[ e^{-(\beta+iw)t} \right]_0^{\infty}$$
$$= -\frac{1}{\beta+iw} [0-1]$$
$$= \frac{1}{\beta+iw}$$

22. State any properties of Linear time invariant system.

1. If  $X(t)$  is WSS process, then  $Y(t)$  is also WSS process.

If the input  $x(t)$  and its output  $y(t)$  are related by  $y(t) = \int_{-\infty}^{\infty} h(u) x(t-u) du$ , then the system is linear time invariant system.