1. A random process $x(t) = A\cos(\omega_0 t + \phi)$ where ω_0 & A are constants and ϕ is independent random variable and ϕ is uniform in $(-\pi, \pi)$. Check whether the process is ergodic (or) not.

Soln.

Given $x(t) = A\cos(\omega_0 t + \phi)$

To prove $x(t)$ is ergodic.

i). Time Average: $\overline{X_T} = \frac{1}{2T}\int_{-T}^{T} x(t)\,dt$

$$= \frac{1}{2T}\int_{-T}^{T} A\cos(\omega_0 t + \phi)\,dt$$

$$= \frac{A}{2T}\left[\frac{\sin(\omega_0 t + \phi)}{\omega_0}\right]_{-T}^{T}$$

$$= \frac{A}{2T\omega_0}\left[\sin(\omega_0 T + \phi) - \sin(-\omega_0 T + \phi)\right]$$

$$= \frac{A}{2T\omega_0}\left[\sin(2\pi + \phi) - \sin(-2\pi + \phi)\right]$$

$\because \omega_0 = \frac{2\pi}{T}$ is angular velocity

$$= \frac{A}{2T\omega_0}\left[\sin(2\pi + \phi) + \sin(2\pi - \phi)\right]$$

$\sin(360 + \theta) = \sin\theta$

$$= \frac{A}{2T\omega_0}\left[\sin\phi - \sin\phi\right]$$

$\sin(360 - \theta) = -\sin\theta$

$\overline{X_T} = 0 \rightarrow (1)$

ii). Ensemble Average :

$$E[x(t)] = E[A\cos(\omega_0 t + \phi)]$$

$$= \int_{-\pi}^{\pi} A\cos(\omega_0 t + \phi)\frac{1}{2\pi}\,d\phi \qquad \because f(\phi) = \frac{1}{b-a} = \frac{1}{2\pi}$$

$$= \frac{A}{2\pi}\int_{-\pi}^{\pi} \cos(\omega_0 t + \phi)\,d\phi$$

$$= \frac{A}{2\pi}\left[\sin(\omega_0 t + \phi)\right]_{-\pi}^{\pi}$$

$$= \frac{A}{2\pi}\left[\sin(\omega_0 t + \pi) - \sin(\omega_0 t - \pi)\right]$$

$$= \frac{A}{2\pi}\left[\sin(\pi + \omega_0 t) + \sin(\pi - \omega_0 t)\right]$$

$$= \frac{A}{2\pi}\left[-\sin\omega_0 t + \sin\omega_0 t\right]$$

$$E[x(t)] = 0 \longrightarrow (2)$$

From (1) & (2)

Time Average = Ensemble Average

Hence the process is ergodic process