Poisson Process :

If $X(t)$ represents the number of occurrences of certain event in $(0, t)$, then the discrete random process $X(t)$ is called the poisson process provided the following postulates are satisfied.

i). $P[1 \text{ occurrences in } (t, t+\Delta t)] = \lambda \Delta t$

ii). $P[0 \text{ occurrences in } (t, t+\Delta t)] = 1 - \lambda \Delta t$

iii). $P[2 \text{ or more occurrences in } (t, t+\Delta t)] = 0$

iv). $X(t)$ is independent of the number of occurences of the event in any interval prior and after the interval $(0, t)$

v). The probability that the event occurs a specified number of times in (t_0, t_0+t) depends only on t but not on t_0

~~Result~~ :

Probability law of Poisson Process :

$$P_X(t) = \frac{e^{-\lambda t} (\lambda t)^x}{x!}, \quad x = 0, 1, 2, \ldots$$

MGF : $e^{\lambda t (e^t - 1)}$

mean : λt $\qquad$ $E[X^2(t)] = \lambda^2 t^2 + \lambda t$

Variance : λt

Properties :

Prob. I. poisson process is a Markov process

Prop. 2 :

Sum of two independent poisson process is a poisson process.

Proof :

MGF of poisson process:

$$M_{X_1}(t) = e^{\lambda_1 t (e^t - 1)}$$

and $M_{X_2}(t) = e^{\lambda_2 t (e^t - 1)}$

Now,

$$M_{X_1 + X_2}(t) = M_{X_1}(t) \cdot M_{X_2}(t) \quad [\because X_1 \text{ \& } X_2 \text{ are independent}]$$

$$= e^{\lambda_1 t (e^t - 1)} \; e^{\lambda_2 t (e^t - 1)}$$

$$= e^{\lambda_1 t (e^t - 1) + \lambda_2 t (e^t - 1)}$$

$$= e^{(\lambda_1 + \lambda_2)(e^t - 1) t}$$ which is the MGF of poisson process with $(\lambda_1 + \lambda_2) t$ as parameter

Hence the sum of two independent poisson process is a poisson process.

Prop 3 :

Difference of two independent poisson process is not a poisson process.

Proof :

Let $X(t) = X_1(t) - X_2(t)$

$$E[X(t)] = E[X_1(t) - X_2(t)] = E[X_1(t)] - E[X_2(t)]$$

$$= \lambda_1 t - \lambda_2 t$$

$$= (\lambda_1 - \lambda_2) t$$

and $E[X^2(t)] = E[\{X_1(t) - X_2(t)\}^2]$

$$= E[X_1^2(t) + X_2^2(t) - 2X_1(t) X_2(t)]$$

$= E[x_1^2(t)] + E[x_2^2(t)] - E[2x_1(t)x_2(t)]$

$= \lambda_1^2 t^2 + \lambda_1 t + \lambda_2^2 t^2 + \lambda_2 t - 2E[x_1(t)]E[x_2(t)]$

$= \lambda_1^2 t^2 + \lambda_1 t + \lambda_2^2 t^2 + \lambda_2 t - 2\lambda_1 t(\lambda_2 t)$

$= \lambda_1^2 t^2 + \lambda_2^2 t^2 - 2\lambda_1 \lambda_2 t^2 + \lambda_1 t + \lambda_2 t$

$= (\lambda_1^2 + \lambda_2^2 - 2\lambda_1 \lambda_2) t^2 + (\lambda_1 + \lambda_2) t$

$E[x^2(t)] = (\lambda_1 - \lambda_2)^2 t^2 + (\lambda_1 + \lambda_2) t$

$\therefore$ $x_1(t) - x_2(t)$ is not a poisson process.