(24)

INTERPOLATION WITH EQUAL INTERVALS

Newton's forward interpolation formula for equal intervals:

Let $x_0, x_1, \cdots x_n$ be equidistant values of x and $y_0, y_1, \cdots y_n$ be the corresponding values of $y = f(x)$.
Let $h = x_i - x_{i-1}$, $i = 1, 2, \cdots n$. Then

$$y = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots + \frac{u(u-1)\cdots(u-(n-1))}{n!}\Delta^n y_0$$

where $u = \frac{x - x_0}{h}$.

(25)

Newton's backward interpolation formula for equal intervals:

Let $x_0, x_1, x_2, \cdots x_n$ be equidistant values of x and $y_0, y_1, y_2, \cdots y_n$ be the corresponding values of $y = f(x)$.
Let $h = x_i - x_{i-1}$, $i = 1, 2, \cdots n$. Then

$$y = y_n + \frac{v}{1!}\nabla y_n + \frac{v(v+1)}{2!}\nabla^2 y_n + \frac{v(v+1)(v+2)}{3!}\nabla^3 y_n + \cdots + \frac{v(v+1)\cdots(v+n-1)}{n!}\nabla^n y_n$$

where $v = \frac{x - x_n}{h}$

Problems :

(1) Find the values of y at $x = 21$ and $x = 28$ from the following data :

x :	20	23	26	29
y :	0.3420	0.3907	0.4384	0.4848

Soln:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
20	0.3420			
		0.0487		
23	0.3907		-0.001	
		0.0477		-0.0003
26	0.4384		-0.0013	
		0.0464		
29	0.4848			

Here $h = 3$

Since $x = 21$ is nearer to the beginning of the table, we use Newton's forward formula.

$$y(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0$$

where $p = \dfrac{x - x_0}{h} = \dfrac{21 - 20}{3} = 0.3333$

$p = 0.3333$

$$y(21) = 0.3420 + (0.3333)(0.0481) + \frac{(0.3333)(-0.6667)(-0.001)}{2} + \frac{(0.3333)(-0.6667)(-1.6667)(-0.0003)}{6}$$

$$= 0.3480 + 0.0162 + 0.0001 - 0.0000185$$

$$\boxed{y(21) = 0.3583}$$

Since $x = 28$ is nearer to the end value, we use Newton's backward interpolation formula,

$$y(x) = y_n + \nabla y_n \cdot v + \frac{v(v+1)}{2!}\nabla^2 y_n + \frac{v(v+1)(v+2)}{3!}\nabla^3 y_n + \cdots$$

where $v = \frac{x - x_n}{h} = \frac{28 - 29}{3}$

$x = 28$

$x_n = 29$

$v = -0.3333$

$$y(28) = 0.4848 + (-0.3333)(0.0464) + \frac{(-0.3333)(0.6667)(-0.0013)}{2} + \frac{(-0.3333)(0.6667)(1.6667)(-0.0003)}{6}$$

$y(28) = 0.4848 - 0.01547 + 0.00014 + 0.00002$

$\boxed{y(28) = 0.4695}$

(2) From the following table of half-yearly premium for policies maturing at different ages, estimate the Premium for policies maturing at age 46 and 63

Age x :	45	50	55	60	65
Premium y :	114.85	96.16	83.32	74.48	68.48

Soln:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
45	114.85				
		-18.69			
			5.85		
50	96.16			-1.85	
		-12.84			0.69
55	83.32		4		
		-8.84		-1.16	
60	74.48		2.84		
		-6			
65	68.48				

Here $h = 5$.

Since $x = 46$ is nearer to the beginning of the table, we use Newton's forward difference formula.

$$y(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0$$

$$+ \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0$$

where $p = \frac{x - x_0}{h} = \frac{46-45}{5} = 0.2$

$p = 0.2$

$$\therefore y(46) = 114.85 + (0.2)(-18.69) + \frac{(0.2)(0.2+1)}{2}(5.85)$$

$$+ \frac{(0.2)(0.2-1)(0.2-2)}{6}(-1.85) +$$

$$\frac{(0.2)(0.2-1)(0.2-2)(0.2-3)}{24}(0.69)$$

$$= 114.85 - 3.738 - 0.468 - 0.0888 - 0.0232$$

$$\boxed{y(46) = 110.532}$$

Since $x = 63$ is nearer to $x = 65$, we use Newton's backward formula

i.e., $y(x) = y_n + q\nabla y_n + \frac{q(q+1)}{2!}\nabla^2 y_n +$

$$\frac{q(q+1)(q+2)}{3!}\nabla^3 y_n + \frac{q(q+1)(q+2)(q+3)}{4!}\nabla^4 y_n$$

28

$$v = \frac{x - x_n}{h} = \frac{63 - 65}{5} = -0.4$$

$$y(63) = 68.48 + (-0.4)(-6) + \frac{(-4)(.6)(2.84)}{2} + \frac{(-0.4)(0.6)(1.6)}{6}(-1.16) + \frac{(-0.4)(0.6)(1.6)(2.6)(.69)}{24}$$

$$= 68.48 + 2.4 - 0.3408 + 0.07424 - .0287$$

$$\boxed{y(63) = 70.5847}$$

(3) From the following table find the value of $\tan 45^\circ 15'$

x° :	45	46	47	48	49	50
$\tan x^\circ$:	1.0000	1.03553	1.07237	1.11061	1.15037	1.19175

Soln :

x°	$y = \tan x^\circ$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
45	1.0000					
		0.03553				
			0.00131			
46	1.03553			0.00009		
		0.03684			0.00003	
			0.0014			−0.00005
47	1.07237			0.00012		
		0.03824			−0.00002	
			0.00152			
48	1.11061			0.0001		
		0.03976				
			0.00162			
49	1.15037					
		0.04138				
50	1.19175					

1 degree $(1^\circ) = 60$ min $(60')$

Here $h = 1^\circ$

1 minute $(1') = 60$ sec $(60'')$

Since $x = 45^\circ 15'$ is nearer to the beginning of the table, we use Newton's forward difference formula.

$$y(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots$$

$$p = \frac{x - x_0}{h} \qquad x_0 = 45^\circ, \; x = 45^\circ 15', \; h = 1^\circ$$

$$= \frac{45^\circ 15' - 45^\circ}{1^\circ} = \frac{15'}{1^\circ} = \frac{15'}{60'} = 0.25 \qquad 1^\circ = 60'$$

$$\boxed{p = 0.25}$$

$$\therefore y(45^\circ 15') = 1 + (0.25)(0.0355\,3) + \frac{(0.25)(-0.75)(0.00131)}{2} +$$

$$\frac{(0.25)(-0.75)(-1.75)}{6}(0.00009) + \frac{(0.25)(-0.75)(-1.75)(-2.75)(0.00003)}{24}$$

$$+ \frac{(0.25)(-0.75)(-1.75)(-2.75)(-3.75)}{120}(-0.00005)$$

$$= 1 + 0.00888 - 0.00012 + 0.000005 - 0.000001 - 0.000001$$

$$\boxed{y(45^\circ 15') = 1.008763}$$

(4) The population of a town is as follows:

Year	1941	1951	1961	1971	1981	1991
Population in Lakhs	20	24	29	36	46	51

Estimate the population increase during the period 1946 to 1976

Soln:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
1941	20					
		4				
1951	24		1			
		5		1		
1961	29		2		0	
		7		1		-9
1971	36		3		-9	
		10		-8		
1981	46		-5			
		5				
1991	51					

Here $h = 10$.

Since $x = 1946$ is nearer to the beginning of the table, we use Newton's forward difference table formula.

$$y(x) = y_0 + p\,\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)\Delta^3 y_0}{3!} + \cdots$$

where $p = \dfrac{x - x_0}{h} = \dfrac{1946 - 1941}{10} = \dfrac{1}{2} = 0.5$

$\boxed{p = 0.5}$

$$y(1946) = 20 + (0.5)4 + \frac{(0.5)(-0.5)}{2} +$$

$$\frac{(0.5)(-0.5)(-1.5)}{6} + \frac{(0.5)(-0.5)(-1.5)(-2.5)(0)}{24}$$

$$+ \frac{(0.5)(-0.5)(-1.5)(-2.5)(-3.5)(-9)}{120}$$

$$= 20 + 2 - 0.125 + 0.0625 - 0.2461$$

$$\boxed{y(1946) = 21.6914}$$

We use backward difference formula to find $y(1976)$.

$$y(x) = y_n + v \nabla y_n + \frac{v(v+1)}{2!} \nabla^2 y_n + \cdots$$

where $v = \frac{x - x_n}{h} = \frac{1976 - 1991}{10} = -1.5$

$$\boxed{v = -1.5}$$

$$y(1976) = 51 + (-1.5)(5) + \frac{(-1.5)(-0.5)(-5)}{2}$$

$$+ \frac{(-1.5)(-0.5)(0.5)(-8)}{6} + \frac{(-1.5)(-0.5)(0.5)(1.5)(-9)}{24}$$

$$+ \frac{(-1.5)(-0.5)(0.5)(1.5)(2.5)(-9)}{120}$$

$$y(1976) = 51 - 7.5 - 1.875 - 0.5 - 0.2109 - 0.1055$$

$$\boxed{y(1976) = 40.8086}$$

Increase in population during the period 1946 to 1976 is $= 40.8086 - 21.6914 = 19.1172$ lakhs.

(5) From the following table, find θ at $x = 43$ and $x = 84$.

x :	40	50	60	70	80	90
θ :	184	204	226	250	276	304

Soln :

x	θ	$\Delta\theta$	$\Delta^2\theta$	$\Delta^3\theta$
40	184			
		20		
50	204		2	
		22		0
60	226		2	
		24		0
70	250		2	
		26		0
80	276		2	
		28		
90	304			

Here $h = 10$.

To find $x = 43$, let us use Newton's forward difference formula.

$$\theta(x) = \theta_0 + p\,\Delta\theta_0 + \frac{p(p-1)}{2}\Delta^2\theta_0 + \cdots$$

where $p = \dfrac{x - x_0}{h} = \dfrac{43-40}{10} = 0.3$

$$\boxed{p = 0.3}$$

$$\theta(43) = 184 + (0.3)(20) + \frac{(0.3)(-0.7)(2)}{2}$$

$$= 184 + 6 - 0.21$$

$$\boxed{\theta(43) = 189.79}$$

To find $x = 84$, let us use Newton's Backward difference formula.

$$\theta(x) = \theta_n + v\,\nabla\theta_n + \frac{v(v+1)}{2!}\nabla^2\theta_n + \cdots$$

where $v = \dfrac{x - x_n}{h} = \dfrac{84-90}{10} = \dfrac{-6}{10} = -0.6$

$$\boxed{v = -0.6}$$

$$\therefore \theta(84) = 304 + (-0.6)(28) + \frac{(-0.6)(0.4)(2)}{2}$$

$$= 304 - 16.8 - 0.24$$

$$= 286.96$$

$$\boxed{\theta(84) = 286.96}$$