Inverse of a matrix - Gauss Jordan method

Let us find the inverse of a non-singular square matrix A of order 3.

If X is the inverse of A, then AX = I where I is the unit matrix of order 3.

Now, we've to find the elements of X

Let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ and $X = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}$

Therefore AX = I reduces to

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \rightarrow (1)$$

which is equivalent to

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x_{11} \\ x_{21} \\ x_{31} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \rightarrow (2)$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x_{12} \\ x_{22} \\ x_{32} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \rightarrow (3)$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x_{13} \\ x_{23} \\ x_{33} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \rightarrow (4)$$

We can solve these eqns. by Gauss Jordan method.

1. Find the inverse of the matrix $\begin{pmatrix} 5 & -2 \\ 3 & 4 \end{pmatrix}$ by Gauss Jordan method.

Soln. :

Let $A = \begin{pmatrix} 5 & -2 \\ 3 & 4 \end{pmatrix}$ and $X = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$ be the inverse of A, so that $AX = I$

The augmented matrix is,

$$[A, I] \sim \left(\begin{array}{cc|cc} 5 & -2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{cc|cc} 1 & -2/5 & 1/5 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right) \quad R_1 \to R_1/5$$

$$\sim \left(\begin{array}{cc|cc} 1 & -2/5 & \frac{1}{5} & 0 \\ 0 & 26/5 & -3/5 & 1 \end{array}\right) \quad R_2 \to R_2 - 3R_1$$

$$\sim \left(\begin{array}{cc|cc} 1 & -2/5 & 1/5 & 0 \\ 0 & 1 & -3/26 & 5/26 \end{array}\right) \quad R_2 \to R_2 \times \frac{5}{26}$$

$$\sim \left(\begin{array}{cc|cc} 1 & 0 & 2/13 & 1/13 \\ 0 & 1 & -3/26 & 5/26 \end{array}\right) \quad R_1 \to R_1 + \left(\frac{2}{5}\right) R_2$$

Hence the inverse of the given matrix is,

$$\begin{bmatrix} 2/13 & 1/13 \\ -3/26 & 5/26 \end{bmatrix} = \frac{1}{26}\begin{bmatrix} 4 & 2 \\ -3 & 5 \end{bmatrix}$$

2). Find the inverse of $\begin{pmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{pmatrix}$ using Gauss-Jordan method.

Soln.:

The augmented matrix is,

$$(A, I) \sim \left(\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 1 & 3 & -3 & 0 & 1 & 0 \\ -2 & -4 & -4 & 0 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 2 & -6 & -1 & 1 & 0 \\ 0 & -2 & 2 & 2 & 0 & 1 \end{array}\right) \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 + 2R_1 \end{array}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 2 & -6 & -1 & 1 & 0 \\ 0 & 0 & -4 & 1 & 1 & 1 \end{array}\right) R_3 \to R_3 + R_2$$

$$\sim \left(\begin{array}{ccc|ccc} 2 & 4 & 0 & 1 & 1 & 0 \\ 0 & 2 & -6 & -1 & 1 & 0 \\ 0 & 0 & -4 & 1 & 1 & 1 \end{array}\right) R_1 \to 2R_1 + R_2$$

$$\sim \left(\begin{array}{ccc|ccc} 2 & 4 & 0 & 1 & 1 & 0 \\ 0 & -8 & 0 & 10 & 2 & 6 \\ 0 & 0 & -4 & 1 & 1 & 1 \end{array}\right) R_2 \to -4R_2 + 6R_3$$

$$\sim \left(\begin{array}{ccc|ccc} 4 & 0 & 0 & 12 & 4 & 6 \\ 0 & -8 & 0 & 10 & 2 & 6 \\ 0 & 0 & -4 & 1 & 1 & 1 \end{array}\right) R_1 \to 2R_1 + R_2$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 1 & 3/2 \\ 0 & 1 & 0 & -5/4 & -1/4 & -3/4 \\ 0 & 0 & 1 & -1/4 & -1/4 & -1/4 \end{array}\right) \begin{array}{l} R_1 \to R_1/4 \\ R_2 \to R_2/-8 \\ R_3 \to R_3/-4 \end{array}$$

Hence the inverse of the given matrix is,

$$\begin{pmatrix} 3 & 1 & 3/2 \\ -5/4 & -1/4 & -3/4 \\ -1/4 & -1/4 & -1/4 \end{pmatrix}$$

3] Find the inverse of the matrix $\begin{pmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & -6 & -7 \end{pmatrix}$ using Gauss Jordan method.

Soln. : The augmented matrix is,

$$[A, I] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 3 & 4 & 5 & 0 & 1 & 0 \\ 0 & -6 & -7 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 4 & 8 & -3 & 1 & 0 \\ 0 & -6 & -7 & 0 & 0 & 1 \end{array}\right] \quad R_2 \to R_2 - 3R_1$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 4 & 8 & -3 & 1 & 0 \\ 0 & 1 & 7/6 & 0 & 0 & -1/6 \end{array}\right] \quad R_3 \to R_3/-6$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 7/6 & 0 & 0 & -1/6 \\ 0 & 0 & 8 & -3 & 1 & 0 \end{array}\right] \quad R_2 \leftrightarrow R_3$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 7/6 & 0 & 0 & -1/6 \\ 0 & 0 & 1 & -3/8 & 1/8 & 0 \end{array}\right] \quad R_1 \to \frac{R_3}{8}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 7/16 & -7/48 & -1/6 \\ 0 & 0 & 1 & -3/8 & 1/8 & 0 \end{array}\right] \quad R_2 \to R_2 - \frac{7}{6}R_3$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 5/8 & 1/8 & 0 \\ 0 & 1 & 0 & 7/16 & -7/48 & -1/6 \\ 0 & 0 & 1 & -3/8 & 1/8 & 0 \end{array}\right]$$

Hence the inverse of the gvn. matrix is

$$\begin{bmatrix} 5/8 & 1/8 & 0 \\ 7/16 & -7/48 & -1/6 \\ -3/8 & 1/8 & 0 \end{bmatrix}$$

$$\begin{pmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{pmatrix} \quad \begin{pmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{pmatrix} \quad \begin{pmatrix} 1/2 & -1/2 & 1/2 \\ -4 & 3 & -1 \\ 5/2 & -3/2 & 1/2 \end{pmatrix}$$

4]. Find the inverse of the matrix $\begin{bmatrix} 2 & 1 & 1 \\ 3 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}$ using Gauss Jordan method.

Soln. The augmented matrix is,

$$[A, I] \sim \left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 3 & 2 & 3 & 0 & 1 & 0 \\ 1 & 4 & 9 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 4 & 9 & 0 & 0 & 1 \\ 3 & 2 & 3 & 0 & 1 & 0 \\ 2 & 1 & 1 & 1 & 0 & 0 \end{array}\right] \quad R_1 \leftrightarrow R_3$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 4 & 9 & 0 & 0 & 1 \\ 0 & -10 & -24 & 0 & 1 & -3 \\ 0 & -7 & -17 & 1 & 0 & -2 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 2R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 4 & 9 & 0 & 0 & 1 \\ 0 & -10 & -24 & 0 & 1 & -3 \\ 0 & 0 & 2 & -10 & 7 & -1 \end{array}\right] \quad R_3 \to 7R_2 - 10R_3$$

$$\sim \left[\begin{array}{ccc|ccc} 2 & 8 & 0 & 90 & -63 & 11 \\ 0 & -10 & 0 & -120 & 85 & -15 \\ 0 & 0 & 2 & -10 & 7 & -1 \end{array}\right] \quad \begin{array}{l} R_1 \to 2R_1 - 9R_2 \\ R_2 \to R_2 + 12R_3 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 20 & 0 & 0 & -60 & 50 & -10 \\ 0 & -10 & 0 & -120 & 85 & -15 \\ 0 & 0 & 2 & -10 & 7 & -1 \end{array}\right] \quad R_1 \to 10R_1 + 8R_2$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -3 & 5/2 & -1/2 \\ 0 & 1 & 0 & 12 & -17/2 & 3/2 \\ 0 & 0 & 1 & -5 & 7/2 & -1/2 \end{array}\right] \quad \begin{array}{l} R_1 \to R_1/20 \\ R_2 \to R_2/-10 \\ R_3 \to R_3/2 \end{array}$$

Hence the inverse of the given matrix is

$$\begin{bmatrix} -3 & 5/2 & -1/2 \\ 12 & -17/2 & 3/2 \\ -5 & 7/2 & -1/2 \end{bmatrix}$$

5]. Find the inverse of the matrix $\begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$ using Gauss Jordan method.

Soln.

$$[A, I] \sim \left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 1 & 4 & 3 & 0 & 1 & 0 \\ 1 & 3 & 4 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 3 & 0 & 4 & 0 & -3 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right] \quad R_1 \to R_1 - 3R_3$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 7 & -3 & -3 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right] \quad R_1 \to R_1 - 3R_2$$

Hence the inverse of the given matrix is

$$\begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

6]. Find A^{-1}, if $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$

Soln. :

$$[A, I] \sim \left[\begin{array}{ccc|ccc} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 3 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 3 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \quad R_1 \leftrightarrow R_2$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & -5 & -8 & 0 & -3 & 1 \end{array}\right] \quad R_3 \to R_3 - 3R_1$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 5 & -3 & 1 \end{array}\right] \quad R_3 \to R_3 + 5R_2$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 5/2 & -3/2 & 1/2 \end{array}\right] \quad R_3 \to R_3/2$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & \frac{-15}{2} & \frac{11}{2} & -3/2 \\ 0 & 1 & 0 & -4 & 3 & -1 \\ 0 & 0 & 1 & 5/2 & -3/2 & 1/2 \end{array}\right] \quad \begin{array}{l} R_1 \to R_1 - 3R_3 \\ R_2 \to R_2 - 2R_3 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & -1/2 & 1/2 \\ 0 & 1 & 0 & -4 & 3 & -1 \\ 0 & 0 & 1 & 5/2 & -3/2 & 1/2 \end{array}\right] \quad R_1 \to R_1 - 2R_2$$

$$\therefore A^{-1} = \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ -4 & 3 & -1 \\ 5/2 & -3/2 & 1/2 \end{bmatrix}$$