Iterative methods:

i). Gauss – Jacobi method

ii). Gauss – Seidel method

Diagonally Dominant:

If the numerical value of the leading diagonal dominant if the numerical value of the leading diagonal element in each row, is greater than or equal to the sum of the numerical values of the other elements in that row.

Gauss Seidel method

Let the system of simultaneous eqns. be

$$\left.\begin{aligned} a_1 x + b_1 y + c_1 z &= d_1 \\ a_2 x + b_2 y + c_2 z &= d_2 \\ a_3 x + b_3 y + c_3 z &= d_3 \end{aligned}\right\} \rightarrow (1)$$

Assume:

$$|a_1| > |b_1| + |c_1|$$

$$|b_2| > |a_2| + |c_2|$$

$$|c_3| > |a_3| + |b_3|$$

The diagonals elements should be dominant, so that the iteration process can be applied. This system of eqns. can also be written as,

$$x = \frac{1}{a_1}(d_1 - b_1 y - c_1 z)$$

$$y = \frac{1}{b_2}(d_2 - a_2 x - c_2 z)$$

$$z = \frac{1}{c_3}(d_3 - a_3 x - b_3 y)$$

1st Iteration: Let $y^{(0)} = z^{(0)} = 0$

$$x^{(1)} = \frac{1}{a_1}(d_1 - b_1 y^{(0)} - c_1 z^{(0)})$$

$$y^{(1)} = \frac{1}{b_2}(d_2 - a_2 x^{(1)} - c_2 z^{(0)})$$

$$z^{(1)} = \frac{1}{c_3}(d_3 - a_3 x^{(1)} - b_3 y^{(1)})$$

2^{nd} Iteration :

$$x^{(2)} = \frac{1}{a_1}(d_1 - b_1 y^{(1)} - c_1 z^{(1)})$$

$$y^{(2)} = \frac{1}{b_2}(d_2 - a_2 x^{(2)} - c_2 z^{(1)})$$

$$z^{(2)} = \frac{1}{c_3}(d_3 - a_3 x^{(2)} - b_3 y^{(2)})$$

This process is repeated till the difference between two consecutive approximations is negligible.

J. Solve by Gauss seidel method

$$x + y + 54z = 110$$
$$27x + 6y - 5z = 85$$
$$6x + 15y + 2z = 72$$

Soln.

Let us rearrange the equations

$$27x + 6y - 5z = 85 \rightarrow (1)$$
$$6x + 15y + 2z = 72 \rightarrow (2)$$
$$x + y + 54z = 110 \rightarrow (3)$$

$\therefore |27| > |6| + |5|$

$|15| > |6| + |2|$

$|54| > |1| + |1|$

$$(1) \Rightarrow \quad x = \frac{85 - 6y + 5z}{27}$$

$$(2) \Rightarrow \quad y = \frac{72 - 6x - 2z}{15}$$

$$(3) \Rightarrow \quad z = \frac{110 - x - y}{54}$$

Let $y_0 = z_0 = 0$.

1st Iteration:

$$x_1 = \frac{1}{27}(85 - 6y_0 + 5z_0) = \frac{1}{27}(85-0-0) = 3.148$$

$$y_1 = \frac{1}{15}(72 - 6x_1 - 2z_0) = \frac{1}{15}(72-6(3.148)-0)$$

$$= 3.541$$

$$z_1 = \frac{1}{54}(110 - x_1 - y_1) = \frac{1}{54}(110-3.148-3.541)$$

$$= 1.913$$

2nd Iteration:

$$x_2 = \frac{1}{27}(85-6(3.541)+5(1.913))$$

$$= 2.432$$

$$y_2 = \frac{1}{15}(72-6(2.432)-2(1.913))$$

$$= 3.572$$

$$z_2 = \frac{1}{54}(110-2.432-3.572)$$

$$= 1.926$$

3rd Iteration:

$$x_3 = \frac{1}{27}(85-6(3.572)+5(1.926))$$

$$= 2.426$$

$$y_3 = \frac{1}{15}(72-6(2.426)-2(1.926))$$

$$= 3.573$$

$$z_3 = \frac{1}{54}(110-2.462-3.573)$$

$$= 1.926$$

4^{th} Iteration :

$$x_4 = \frac{1}{27}(85 - 6y_3 + 5z_3) = \frac{1}{27}(85 - 6(3.573) + 5(1.926))$$

$$= 2.425$$

$$y_4 = \frac{1}{15}(72 - 6(2.425) - 2(1.926))$$

$$= 3.573$$

$$z_4 = \frac{1}{54}(110 - 2.425 - 3.573)$$

$$= 1.926$$

5^{th} Iteration :

$$x_5 = \frac{1}{27}(85 - 6(3.573) + 5(1.926)) = 2.425$$

$$y_5 = \frac{1}{15}(72 - 6(2.425) - 2(1.926)) = 3.573$$

$$z_5 = \frac{1}{54}(110 - 2.425 - 3.573) = 1.926$$

$\therefore$ The Solution is $x = 2.425$

$y = 3.573$

$z = 1.926$

2] Solve the following system by Gauss Seidel method

$$9x - y + 2z = 9$$
$$x + 10y - 2z = 15$$
$$2x - 2y - 13z = -17$$

Soln.

The given system of equations are

$$9x - y + 2z = 9 \longrightarrow (1)$$
$$x + 10y - 2z = 15 \longrightarrow (2)$$
$$2x - 2y - 13z = -17 \longrightarrow (3)$$

Clearly the coefficient matrix is diagonally dominant, so we can apply Gauss Seidel method.

$(1) \Rightarrow x = \frac{1}{9}(9+y-2z)$

$(2) \Rightarrow y = \frac{1}{10}(15-x+2z)$

$(3) \Rightarrow z = \frac{1}{13}(2x-2y+17)$

1^{st} Iteration:

Let us assume $y^{(0)} = z^{(0)} = 0$

$$x_1 = \frac{1}{9}(9+y_0-2z_0) = \frac{9}{9} = 1$$

$$y_1 = \frac{1}{10}(15-x_1+2z_0) = \frac{1}{10}(15-1+0) = 1.4$$

$$z_1 = \frac{1}{13}(2x_1-2y_1+17) = \frac{1}{13}(2(1)-2(1.4)+17)$$
$$= 1.3626$$

2^{nd} Iteration:

$$x_2 = \frac{1}{9}(9+1.4-2(1.3626)) = 0.8528$$

$$y_2 = \frac{1}{10}(15-0.8528+2(1.3626)) = 1.6872$$

$$z_2 = \frac{1}{13}(2(0.8528)-2(1.6872)) = 1.1795$$

3^{rd} Iteration:

$$x_3 = \frac{1}{9}(9+1.6872-2(1.1793)) = 0.9254$$

$$y_3 = \frac{1}{10}(15-0.9254+2\times1.1793) = 1.6433$$

$$z_3 = \frac{1}{13}(2(0.9254)-2(1.6433)+17) = 1.1972$$

4^{th} Iteration:

$$x_4 = \frac{1}{9}(9+1.6433-2(1.1972)) = 0.9165$$

$$y_4 = \frac{1}{10}(15-0.9165+2(1.1972)) = 1.6478$$

$$z_4 = \frac{1}{13}(2(0.9165)-2(1.6478)+17) = 1.1952$$

5^{th} Iteration :

$$x_5 = \frac{1}{9}(9 + 1.6478 - 2(1.1952)) = 0.9175$$

$$y_5 = \frac{1}{10}(15 - 0.9175 + 2(1.1952)) = 1.6473$$

$$z_5 = \frac{1}{13}(2(0.9175) - 2(1.6473) + 17) = 1.1954$$

6^{th} Iteration :

$$x_6 = \frac{1}{9}(9 + 1.6473 - 2(1.1954)) = 0.9174$$

$$y_6 = \frac{1}{10}(15 - 0.9174 + 2(1.1954)) = 1.6473$$

$$z_6 = \frac{1}{13}(2(0.9174) - 2(1.6473) + 17) = 1.1954$$

$\therefore$ The Solution is $x = 0.9174$, $y = 1.6473$, $z = 1.1954$.