Unit - I

Solutions of Equations and Eigen value problems

* Newton's method (or) Newton Raphson method

Newton Raphson method is extensively used for analysis of flow in water distribution networks.

It is used to find the roots of nonlinear equations.

Formula :

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad \text{provided } f'(x_n) \neq 0.$$

Order = 2

N_R Condition : $|f(x)\, f''(x)| < |f'(x)|^2$

Problems :

I. Find the smallest positive root of the eqn.

$x^3 - 2x + 0.5 = 0$

Soln.

Let $f(x) = x^3 - 2x + 0.5$

Now $f'(x) = 3x^2 - 2$

Now $f(0) = 0.5$

$f(1) = -0.5$

$\therefore$ The root lies between 0 & 1.

Since $|f(0)| = |f(1)|$.

Let us assume $x_0 = 0$.

Newton Raphson Formula :

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Put $n=0$, $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

$$= 0 - \frac{0.5}{-2}$$

$$x_1 = 0.25$$

put $n=1$, $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$$= 0.25 - \frac{f(0.25)}{f'(0.25)}$$

$$= 0.25 - \frac{(0.25)^3 - 2(0.25) + 0.5}{3(0.25)^2 - 2}$$

$$x_2 = 0.2586$$

put $n=2$, $x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$

$$= 0.2586 - \frac{f(0.2586)}{f'(0.2586)}$$

$$= 0.2586 - \frac{(0.2586)^3 - 2(0.2586) + 0.5}{3(0.2586)^2 - 2}$$

$$x_3 = 0.2586$$

Since x_2 & x_3 are equal root, the smallest positive root is 0.2586.

2]. Compute the real root of $x \log x = 1.2$ correct to 3 decimal places using Newton Raphson method

Soln.

Let $f(x) = x \log x - 1.2$

$f'(x) = x\left(\frac{1}{x}\right) + \log x \,(1) - 0 = 1 + \log x$

Now ~~$f(0) = -1.2$~~ (−ve)

$f(1) = -1.2$ (−ve)

$f(2) = -0.5979$ (−ve)

$f(3) = 0.2314$ (+ve)

$\therefore$ The root lies between 2 & 3.

Since $|f(2)| > |f(3)|$. Let us assume $x_0 = 3$

N-R Formula: $x_{n+1} = x_n - \frac{F(x_n)}{F'(x_n)}$

$$x_1 = x_0 - \frac{F(x_0)}{F'(x_0)}$$

$$= 3 - \frac{F(3)}{F'(3)} = 3 - \frac{[3 \log 3 - 1.2]}{1 + \log 3}$$

$$x_1 = 2.8434$$

$$x_2 = x_1 - \frac{F(x_1)}{F'(x_1)}$$

$$= 2.8434 - \frac{[2.8434 \log 2.8434 - 1.2]}{1 + \log 2.8434}$$

$$x_2 = 2.7822$$

$$x_3 = x_2 - \frac{F(x_2)}{F'(x_2)}$$

$$= 2.7822 - \frac{[2.7822 \log 2.7822 - 1.2]}{1 + \log 2.7822}$$

$$x_3 = 2.7576$$

Illy $x_4 = 2.7476$

$x_5 = 2.7435$

$x_6 = 2.7418$

$x_7 = 2.7411$

$x_8 = 2.7408$

$x_9 = 2.7407$

$x_{10} = 2.7407$

$\therefore$ The required root is 2.7406.

Hw Find the +ve root of $2x^3 - 3x - 6 = 0$

Ans: 1.7838

3]. Find the -ve root of $x^3 - \sin x + 1 = 0$ (Radian mode)

Soln. :

Let $f(x) = x^3 - \sin x + 1$

$f'(x) = 3x^2 - \cos x$

Now $f(0) = 1$ (+ve)

$f(-1) = 0.8415$ (+ve)

$f(-2) = -6.0907$ (-ve)

$\therefore$ The root lies between -1 & (-2)

Since $|f(-1)| < |f(-2)|$. Let us assume $x_0 = -1$

Now, $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$

$x_1 = -1.3421$

$x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$

$x_2 = -1.2564$

$x_3 = -1.2491$

$x_4 = -1.2491$

Since x_3 & x_4 are equal.

$\therefore$ The required -ve root is -1.2491

Q. Obtain Newton's Iterative formula for finding $\sqrt{N}$ where N is a +ve real no.. Hence evaluate $\sqrt{5}$.

Soln.

Let $x = \sqrt{N}$

$x^2 = N$

$\Rightarrow x^2 - N = 0$

$f(x) = x^2 - N$

$f'(x) = 2x$

Now $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

$= x_n - \dfrac{x_n^2 - N}{2x_n}$

$= \dfrac{2x_n^2 - x_n^2 + N}{2x_n}$

$= \dfrac{x_n^2 + N}{2x_n}$, which is an iterative formula for $\sqrt{N}$.

To find $\sqrt{5}$.

$x = \sqrt{5}$

$x^2 - 5 = 0$

$\Rightarrow f(x) = x^2 - 5$; $f'(x) = 2x$

$f(0) = -5$ (−ve)

$f(1) = 1 - 5 = -4$ (−ve)

$f(2) = 4 - 5 = -1$ (−ve)

$f(3) = 9 - 5 = 4$ (+ve)

$\therefore$ The root lies between 2 & 3.

Since $|f(2)| < |f(3)|$, let us assume $x_0 = 2$

Now $x_{n+1} = \dfrac{x_n^2 + N}{2x_n}$

Put $n=0$, $x_1 = \dfrac{x_0^2 + 5}{2x_0} = \dfrac{4+5}{2(2)} = \dfrac{9}{4} = 2.25$

$$x_2 = \frac{x_1^2 + N}{2x_1} = \frac{(2.25)^2 + 5}{2(2.25)} = 2.2361$$

$$x_3 = \frac{x_2^2 + 5}{2x_2} = \frac{(2.2361)^2 + 5}{2(2.2361)} = 2.2361$$

$\therefore$ The value of $\sqrt{5} = 2.2361$

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2) Find the iterative formula for finding the value of $\frac{1}{N}$, where N is a real no. using NRM. Hence evaluate $\frac{1}{26}$ correct to 4 decimal places.

Soln.

Let $x = \frac{1}{N}$

i.e., $N = \frac{1}{x}$

Let $f(x) = \frac{1}{x} - N$; $f'(x) = -\frac{1}{x^2}$

Now, $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

$$= x_n - \frac{\frac{1}{x_n} - N}{\frac{-1}{x_n^2}}$$

$$= x_n + x_n^2 \left(\frac{1 - Nx_n}{x_n} \right)$$

$$= x_n + x_n(1 - Nx_n)$$

$$= x_n + x_n - Nx_n^2$$

$x_{n+1} = 2x_n - Nx_n^2$, which is the iterative formula

To find $\frac{1}{26}$; $N = 26$

$f(x) = \frac{1}{x} - 26$; $f'(x) = -\frac{1}{x^2}$

$f(0) = -26$ (-ve)

$f(1) = -25$ (-ve)

$f(2) = -25.5$ (-ve) [It's impossible to find the roots]

Let us take $x_0 = \frac{1}{25} = 0.04$, nearer to the given N

Let $x_0 = 0.04$

WKT, $x_{n+1} = 2x_n - N x_n^2$

$$x_1 = 2x_0 - 26\, x_0^2$$

$$= 2(0.04) - 26(0.04)^2$$

$$x_1 = 0.0384$$

$$x_2 = 0.0384$$

Since x_1 & x_2 are equal, the value of $\frac{1}{26} = 0.0384$

3]. Derive Newton's algorithm for finding the pth root of a number N & find the value of $(24)^{1/3}$.

Soln.

Let $x = N^{\frac{1}{P}}$

$$x^P = N$$

$$\Rightarrow x^P - N = 0$$

Let $f(x) = x^P - N$; $f'(x) = Px^{P-1}$

WKT, $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

$$= x_n - \frac{x_n^P - N}{Px_n^{P-1}}$$

$$x_{n+1} = \frac{Px_n^P - x_n^P + N}{Px_n^{P-1}} = \frac{(P-1)\, x_n^P + N}{Px_n^{P-1}}$$

To find $(24)^{1/3}$

Here $N = 24$, $P = 3$

$f(x) = x^p - N$

$f(x) = x^3 - 24$

$f(0) = -24$ (−ve)

$f(1) = -23$ (−ve)

$f(2) = -16$ (−ve)

$f(3) = 3$ (+ve), the root lies between 2 & 3.

Since $|f(2)| > |f(3)|$, Let us assume $x_0 = 3$.

$$x_{n+1} = \frac{(3-1)x_n^3 + 24}{3x_n^{3-1}} = \frac{2x_n^3 + 24}{3x_n^2}$$

$$x_1 = \frac{2x_0^3 + 24}{3x_0^2} = 2.8888$$

$x_2 = 2.8845$

$x_3 = 2.8844$

$x_4 = 2.8844$

Since $x_3 = x_4$, the required root is 2.8844.