Cauchy's homogoneoous linear eqnation

$$a_0 x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1}y}{dx^{n-1}} + \cdots a_n y = \phi(x) \text{ (or) } Q(x)$$

Put $x = e^t$ (or) $t = \log x$, then if $\theta = \frac{d}{dt}$

$$x \frac{dy}{dx} = xDy = \theta y$$

$$x^2 \frac{d^2y}{dx^2} = x^2 D^2 y = \theta(\theta - 1)y = (\theta^2 - \theta)y$$

$$x^3 \frac{d^3y}{dx^3} = x^3 D^3 y = \theta(\theta - 1)(\theta - 2)y$$

Example 1: Solve $x^2\frac{d^2y}{dx^2}+4x\frac{dy}{dx}+2y=x^2+\frac{1}{x^2}$

Sol

The given eqn is $(x^2D^2+4xD+2)y=x^2+\frac{1}{x^2} \rightarrow (1)$

where $D=\frac{d}{dx}$ which is a Cauchy homogeneous eqn.

Put

$x=e^t$

$xD=\theta$

$x^2D^2=\theta(\theta-1)$

$=\theta^2-\theta$

$x^2+\frac{1}{x^2}=(e^t)^2+\frac{1}{(e^t)^2}$

$=e^{2t}+\frac{1}{e^{2t}}$

$=e^{2t}+e^{-2t}$

$(\theta^2+3\theta+2)y=e^{2t}+e^{-2t}$

which linear eqn. The A.E is $m^2+3m+2=0$

$(m+1)(m+2)=0$

$m=-1,-2$

$C.F=Ae^{-t}+Be^{-2t}$

$PI_1=\frac{1}{\theta^2+3\theta+2}e^{2t}$

$=\frac{1}{2^2+3(2)+2}e^{2t}$

$=\frac{1}{12}e^{2t}$

$P.I_2=\frac{1}{\theta^2+3\theta+2}e^{-2t}=\frac{1}{0}e^{-2t}=\infty$

$=\frac{t}{2\theta+3}e^{-2t}$

$=\frac{t}{-1}e^{-2t}=-te^{-2t}$

The general solution is $y=C.F+P.I$

$y=Ae^{-t}+Be^{-2t}+\frac{1}{12}e^{2t}-te^{-2t}$

$=A(e^t)^{-1}+B(e^t)^{-2}+\frac{1}{12}(e^t)^2-t(e^t)^{-2}$

$=Ax^{-1}+Bx^{-2}+\frac{1}{12}x^2-\log x.x^{-2}$